UMBC — CMSC 203
Spring 2000 — Discrete Structures
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Notes on Recurrences

Linear Homogeneous Recurrences with Constant Coefficients

We begin by recalling some terminology from the text. Roughly, a sequence $\{a_n\}_{n\geq 0}$ satisfies a **recurrence relation** if the term a_n can be computed in terms of $a_0, a_1, \ldots, a_{n-1}$. We say **the recurrence relation has degree** d if a_n can be computed in terms of the previous d terms:

$$a_n = R(a_{n-1}, \ldots, a_{n-d}) \quad \text{for all } n \geq d$$

for some function R. A recurrence relation is **linear** if it is linear in the terms $a_{n-1}, \ldots, a_{n-d}$; that is, the relation has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d} + c,$$

where c and the c_i's are allowed to depend on n, but not on the a_i's. For a linear recurrence, if the c_i's are all constants, we say the recurrence has **constant coefficients**, and if $c = 0$ we say the recurrence is **homogeneous**.

The simplest case is that of a homogeneous linear recurrence relation of degree d, with constant coefficients. We can write the relation as

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_d a_{n-d} = 0 \tag{1}$$

where the c_i's are constants. Let's start by trying to **guess** a solution. Suppose $a_n = t^n$ is a solution to (1) for some constant t. Then (1) implies that

$$t^n - c_1 t^{n-1} - c_2 t^{n-2} - \cdots c_{n-d} t^{n-d} = 0.$$

If we divide both sides by t^{n-d}, we obtain a condition on t which doesn't depend on n:

$$t^d - c_1 t^{d-1} - c_2 t^{d-2} - \cdots c_{n-d} = 0. \tag{2}$$

The polynomial on the left hand side of (2) is called the **characteristic polynomial** of the recurrence relation. We find that $a_n = t^n$ satisfies the recurrence only if t is a root of the characteristic polynomial. (Note that the "degree" of the recurrence is the same as the "degree" of the polynomial.)

Now, how can we find more solutions? This is where the **linearity** of the recurrence comes in. Notice that if a_n and b_n are two sequences which satisfy the recurrence (1), then for any constants A and B, the **linear combination**

$$c_n = Aa_n + Bb_n$$

is another sequence which also satisfies (1)! The game now is to come up with a short list of "basic" solutions to the recurrence (1) such that **any** solution is some linear combination of solutions in our list. (If you're familiar with linear algebra, the set of sequences which satisfy a linear homogeneous recurrence form a vector space; we want to construct a basis.)

How many "basic" solutions should there be in our list? Well, if the recurrence has degree d, so that a_n depends only on $a_{n-1}, \ldots, a_{n-d}$, then we need to specify the first d terms before the recurrence "kicks in." These can be any numbers we like, so we can make essentially d independent choices; we have d "degrees of freedom." On the other hand, how many "degrees of freedom" do we have if we're constructing a linear combination of solutions from our list of "basic" solutions? The answer is just the number of basic solutions, since that's the number of coefficients we get to pick. So, for a degree d recurrence, we'd like to find d "basic" solutions.

In the case of homogeneous linear recurrences with constant coefficients, one can prove the following theorem, which gives us a complete list of basic solutions, from which all solutions can be constructed.

Theorem 1 *Suppose we have a homogeneous linear recurrence with constant coefficients. Let $p(t)$ denote the characteristic polynomial, and suppose that λ is a root of $p(t)$ having multiplicity r; that is,*

$$(t-\lambda)^r \textit{ divides } p(t).$$

1. *Each of the r sequences $\lambda^n, n\lambda^n, n^2\lambda^n, \ldots, n^{r-1}\lambda^n$ satisfies the recurrence.*
2. *(the really cool part)* **Any** *sequence satisfying the recurrence can be written uniquely as a linear combination of solutions constructed in part (1).*

You should keep in mind the "fundamental theorem of algebra," which states that any polynomial $p(t)$, whose leading coefficient is equal to 1, can be factored completely into linear factors of the form $t-\lambda$, provided we allow λ to take complex values. If our recurrence has degree d, then $p(t)$ will have d linear factors, and the total number of "basic solutions" constructed in part (2) of the theorem will be d.

An Example

Let's try an example. Consider the recurrence

$$a_n = 3a_{n-1} - 4a_{n-3},$$

with initial conditions $a_0 = 0$, $a_1 = 1$, $a_2 = 13$. We seek a closed-form expression for a_n.

If we start generating the next few terms, we get $39, 113, 287, 705, 1663, \ldots$ There doesn't seem to be any obvious formula. Nonetheless, we can find the answer as follows: the characteristic polynomial is

$$p(t) = t^3 - 3t^2 + 4 = (t+1)(t-2)^2.$$

This has two roots: -1, with multiplicity one, and 2, with multiplicity 2. By Theorem 1, any sequence satisfying the recurrence is a linear combination of the three sequences $(-1)^n$, 2^n, and $n2^n$. In other words, we have

$$a_n = A(-1)^n + B2^n + Cn2^n \tag{3}$$

for some constants A, B, and C.

How do we determine the constants? We just select them so that the initial conditions are satisfied. We have three initial conditions, for $n = 0, 1, 2$. Plugging each of these values of n in turn into (3) gives three equations involving A, B and C:

$$\begin{array}{rcrl} a_0 = 0 & \Rightarrow & A + B & = 0 \\ a_1 = 1 & \Rightarrow & -A + 2B + 2C & = 1 \\ a_2 = 13 & \Rightarrow & A + 4B + 8C & = 13 \end{array}$$

These equations can easily be solved for A, B and C, for example by successively eliminating variables. We find $A = 1$, $B = -1$ and $C = 2$, giving us a closed-form for our sequence:

$$a_n = (-1)^n - 2^n + 2n2^n = (2n-1)2^n + (-1)^n.$$

Some Non-homogeneous Recurrences

Many non-homogeneous linear recurrence relations can be dealt with easily. We will just do an example here.

$$a_{n+2} = a_{n+1} + a_n + 2^n, \qquad n \geq 0. \tag{4}$$

This is non-homogeneous, due to the 2^n term. We will "homogenize" it by finding a **different** recurrence, also satisfied by the sequence $\{a_n\}$, which **is** homogeneous. One way to do this is to try to use a "slide" of the recurrence to "cancel out" the non-homogeneous part. In our example, we take both the original recurrence, and a slide:

$$\begin{array}{rcl} a_{n+2} & = & a_{n+1} + a_n \quad + 2^n \\ a_{n+1} & = & a_n \quad + a_{n-1} + 2^{n-1} \end{array}$$

Now subtract twice the second equation from the first, so that the $2^{\cdots}$ terms cancel. We find

$$a_{n+2} - 2a_{n+1} = a_{n+1} + a_n - 2a_n - 2a_{n-1},$$

or

$$a_{n+2} = 3a_{n+1} - a_n - 2a_{n-1}. \tag{5}$$

This is a linear homogeneous recurrence which is valid for all $n \geq 1$ (do you see why?) So we're in business; given some initial conditions, we could solve it using the methods we've already developed.

It's worth stopping to smell the roses here. The characteristic polynomial of the recurrence in (5) is $t^3 - 3t^2 + t + 2 = 0$, which factors as $(t^2 - t - 1)(t - 2) = 0$. You should recognize the first factor as the characteristic polynomial of the "homogeneous part" of (4). (That is, if the 2^n term were missing in (4), the resulting homogeneous recurrence would have characteristic polynomial $t^2 - t - 1$.) The second factor, $(t - 2)$, is the characteristic polynomial of the recurrence satisfied by the 2^n term.

This is a special case of something very general. Suppose a sequence a_n satisfies a recurrence which would be a linear homogeneous recurrence with constant coefficients, except for one extra term b_n added in (in the above example, $b_n = 2^n$.) Then a_n does satisfy a linear homogeneous recurrence with constant coefficients, and its characteristic polynomial is the product of two factors: one corresponding to the homogeneous part of the recurrence, and the other corresponding to the "non-homogeneous term" b_n.

We won't prove this result here, since it really belongs in a discussion of vector spaces and linear operators on sequences. But it is useful to know that such a factorization exists, since otherwise it might be harder to guess the factors of the characteristic polynomial.