Homework X
Due: Monday, May 15

§7.2: 3, 13, 16, 28.
§7.4: 13, 24.
§10.2: 5.

And the last problem: consider the finite-state automaton $A$ over the input alphabet $\Sigma = \{0, 1\}$, given by the following annotated next-state table:

\[
\begin{array}{c|cc}
\text{inputs} & 0 & 1 \\
\hline
\circ & s_0 & s_1 \\
s_1 & s_1 & s_0
\end{array}
\]

For any integer $n \geq 0$, let $a_n$ be the number of words of length $n$ over $\Sigma$ which are accepted by $A$; that is, $a_n = |L_A \cap \Sigma^n|$. Compute $a_0$, $a_1$ and $a_2$, and show that $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$. (You don’t have to solve the recurrence unless you really want to.)

[Remark: This is a special case of a very general fact: the number of words of length $n$ accepted by any FSA eventually satisfies a linear recurrence with constant coefficients.]

For problem 24 in section 7.4, you may use the method outlined in the book, or you may use the following theorem which was proved in class.

**Pumping Lemma.** Let $A$ be a finite-state automaton over an alphabet $\Sigma$, having $N$ states. Suppose $w = xyz$ is a word accepted by $A$, where $x$, $y$ and $z$ are words in $\Sigma^*$ and $\text{length}(y) \geq N$. Then there exist words $a$, $b$ and $c$, with $\text{length}(b) \geq 1$, such that $y = abc$, and such that $xab^kcz$ is accepted by $A$ for every integer $k \geq 1$.

(Here’s the idea of the proof. The path corresponding to $y$ passes through at least $N + 1$ states. By the pigeon-hole principle (§7.2), it must pass through some state $s$ twice. Write $y = abc$, where $b$ is the sequence of symbols read between two occurrences of $s$. Then $N^*(s_0, xab^k) = N^*(s_0, xab) = s$ for any $k \geq 1$. It follows that

\[
N^*(s_0, xab^kcz) = N^*(s_0, xabcz) = N^*(s_0, w)
\]

for any $k \geq 1$; the result follows since $N^*(s_0, w)$ is an accept state.)