Section 6: The Algebra of Combinations

• In this section, we will look at some identities involving the choosery function. We will also resurrect Pascal’s Triangle and see how it relates to combinations.

• Three Simple Calculations:

\[ C(n,n) = \frac{n!}{[n!(n-n)!]} = \frac{n!}{(n!0!)} = \frac{n!}{n!} = 1. \]

\[ C(n,1) = \frac{n!}{[1!(n-1)!]} = \frac{n(n-1)!}{(n-1)!} = n. \]

\[ C(n,2) = \frac{n!}{[2!(n-2)!]} = \frac{n(n-1)(n-2)!}{[2(n-2)!]} = \frac{n(n-1)/2.} \]
A Half-empty Set

• We have already seen and used the identity:
  \[ C(n,r) = C(n,n-r). \]

• This identity is easy to demonstrate algebraically.

• However, we can reason it combinatorically.

• We use \( C(n,r) \) to count how many subset of size \( r \) an \( n \)-element set \( A \) has. But we can identify uniquely with any \( r \)-element subset \( B \) of \( A \) the \( (n-r) \)-element subset that is its complement, \( A-B \). Moreover, this identification is a bijection. (Why?) Hence, the number of \( r \)-element subsets equals the number of \( (n-r) \)-element subsets.
Using Substitutions

• We have seen the identity $C(n,2) = n(n-1)/2$.
• Combining this with $C(n,2) = C(n,n-2)$, we see that $C(n,n-2) = n(n-1)/2$.
• We can now use this to get related identities by substituting “interesting” values for $n$:
  - $n \rightarrow n+1$: $C(n+1,n-1) = n(n+1)/2$.
  - $n \rightarrow n-1$: $C(n-1,n-3) = (n-1)(n-2)/2$.
  - $n \rightarrow n+2$: $C(n+2,n) = (n+2)(n+1)/2$. 
Pascal’s Triangle

Recall the number array we call Pascal’s Triangle:

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
```

Rule of generation: $T(n,r) = T(n-1,r-1) + T(n-1,r)$. 
Using Pascal’s Triangle

• One application of Pascal’s Triangle is to find the coefficients of the binomial expansion \((a + b)^n\).
• For example, to expand \((a + b)^5\) we look at the 6th row: 1,5,10,10,5,1 to get:
  \[1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5.\]
• However, the Binomial Theorem tells us each of these coefficients is of the form \(C(5,r)\).
• Thus each term of Pascal’s Triangle is actually a combination, that is \(T(n,r) = C(n,r)\).
Pascal’s Formula

• Now, if we replace the terms of the rule of generation with the binomial coefficients, we get: \( C(n, r) = C(n-1, r-1) + C(n-1, r) \).

• Proof: \( C(n-1, r-1) + C(n-1, r) \)

\[
= (n-1)!/(r-1)!(n-1-r+1)! + (n-1)!/r!(n-r-1)!
\]

\[
= (n-1)!/(r-1)!(n-r)! + (n-1)!/r!(n-r-1)!
\]

\[
[\text{want } (r-1)! \to r! \quad \text{want } (n-r-1)! \to (n-r)!]
\]

\[
= (n-1)!r/r!(n-r)! + (n-1)!(n-r)/r!(n-r)!
\]

\[
= [(n-1)!(r + n - r)] / r!(n-r)!
\]

\[
= (n-1)!n / r(n-r)! = n!/r!(n-r)! = C(n,r).
\]