Section 5: Combinations with Repetition

- In the last section, we saw how to count combinations, where order does not matter, based on permutation counts, and we saw how to count permutations where repetitions occur.
- Now, we shall consider the case where we don’t want order to matter, but we will allow repetitions to occur.
- This will complete the matrix of counting formulae, indexed by order and repetition.

A Motivating Example

- How many ways can I select 15 cans of soda from a cooler containing large quantities of Coke, Pepsi, Diet Coke, Root Beer and Sprite?
- We have to model this problem using the chart:

	Coke	Pepsi	Diet Coke	Root Beer	Sprite	
A:	111	111	111	111	111	=15
B:	11		111111	111111	1	=15
C:		1111	1111111	1111		=15

- Here, we set an order of the categories and just count how many from each category are chosen.

A Motivating Example (*cont'd.*)

- Now, each event will contain fifteen 1's, but we need to indicate where we transition from one category to the next. If we use 0 to mark our transitions, then the events become:

  A: 11101110111011101111

  B: 11001111111011111101

  C: 00111101111111101111

- Thus, associated with each event is a binary string with #1's = #things to be chosen and #0's = #transitions between categories.

Counting Generalized Combinations

- From this example we see that the number of ways to select 15 sodas from a collection of 5 types of soda is C(15 + 4,15) = C(19,15) = C(19,4).
- Note that #zeros = #transitions = #categories – 1.
- Theorem: The number of ways to fill *r* slots from *n* catgories with repetition allowed is:

  $C(r + n - 1, r) = C(r + n - 1, n - 1).$

- In words, the counts are:

  C(#slots + #transitions, #slots)

  or C(#slots + #transitions, #transitions).

Another Example

- How many ways can I fill a box holding 100 pieces of candy from 30 different types of candy?

  Solution: Here #slots = 100, #transitions = 30 – 1, so there are C(100+29,100) = 129!/(100!29!) different ways to fill the box.

- How many ways if I must have at least 1 piece of each type?

  Solution: Now, we are reducing the #slots to choose over to (100 – 30) slots, so there are C(70+29,70) = 99!/70!29!

When to Use Generalized Combinations

- Besides categorizing a problem based on its order and repetition requirements as a generalized combination, there are a couple of other characteristics which help us sort:
  - In generalized combinations, having all the slots filled in by only selections from one category is allowed;
  - It is possible to have more slots than categories.

Integer Solutions to Equations

- One other type of problem to be solved by the generalized combination formula is of the form:

  How many non-negative integer solutions are there to the equation $a + b + c + d = 100$.

- In this case, we could have 100 *a*'s or 99 *a*'s and 1 *b*, or 98 *a*'s and 2 *d*'s, etc.
- We see that the #slots = 100 and we are ranging over 4 categories, so #transitions = 3.
- Therefore, there are C(100+3,100) = 103!/100!3! non-negative solutions to $a + b + c + d = 100$.

Integer Solutions with Restrictions

- How many integer solutions are there to:
  $$a + b + c + d = 15,$$
  when $a \geq 3$, $b \geq 0$, $c \geq 2$ and $d \geq 1$?
- Now, solution "strings" are $111a0b011c01d$, where the a,b,c,d are the remaining numbers of each category to fill in the remaining slots.
- However, the number of slots has effectively been reduced to 9 after accounting for a total of 6 restrictions.
- Thus there are C(9+3,9) = 12!/(9!3!) solutions.

More Integer Solutions & Restrictions

- How many integer solutions are there to:

  $$a + b + c + d = 15,$$

  when $a \geq -3$, $b \geq 0$, $c \geq -2$ and $d \geq -1$?
- In this case, we alter the restrictions and equation so that the restrictions "go away." To do this, we need each restriction ≥ 0 and balance the number of slots accordingly.
- Hence $a \geq -3+3$, $b \geq 0$, $c \geq -2+2$ and $d \geq -1+1$, yields $a + b + c + d = 15+3+2+1 = 21$
- So, there are C(21+3,21) = 24!/(21!3!) solutions.

Summary

- Theorem: The number of integer solutions to:

  $$a_1 + a_2 + a_3 + ... + a_n = r,$$

  when $a_1 \geq b_1$, $a_2 \geq b_2$, $a_3 \geq b_3$, ..., $a_n \geq b_n$ is $C(r+n-1-b_1-b_2-b_3-...-b_n , r-b_1-b_2-b_3-...-b_n)$.
- Theorem: The number of ways to select r things from n categories with b total restrictions on the r things is $C(r + n - 1 - b , r - b)$.
- Corollary: The number of ways to select r things from n categories with at least 1 thing from each category is $C(r - 1 , r - n)$ (set $b = n$).