Section 2: Mathematical Induction

- Principle of Mathematical Induction;
- Formula for the Sum of the First $n$ Integers;
- Formula for the Sum of a Geometric Series.
What Is Induction

• One of the more recently developed proof techniques;
• Used to verify conjectures about processes that occur repeatedly, according to definite patterns;
• Used to prove statements indexed on the Natural Numbers (i.e. For all integers $n \geq 0$, ...)
Climbing Infinite Staircases

- Consider the problem of trying to climb an infinitely tall staircase.
- How can I know that this staircase is climbable?
- First, show the staircase exists. (*Basis*)
- Second, show standing at any arbitrary step implies I can climb to the next step. (*Induction*)
- The *basis* shows there is an initial step.
- The *induction* validates a rule of motion from one step to another.
Principle of Mathematical Induction

• Let $P(n)$ be a predicate that is defined for integers $n$, and let $a$ be a fixed integer.

• Suppose the following two statements are true:
  1. $P(a)$ is true;
  2. For all integers $k \geq a$, if $P(k)$ is true, then $P(k + 1)$ is true.

• Then, for all integers $n \geq a$, $P(n)$ is true.
Outline of an Inductive Proof

• Basis step says to show $P(\text{initial value})$ is true.
• Inductive step says:
  – Assume $P(k)$ is true for an arbitrary $k$.
  – Use this to show $P(k + 1)$ is true.
• This assumption of $P(k)$ is called the \textit{inductive hypothesis}.
Tuppence and Nickels

• Show that any monetary value of 4 cents or more can be composed of tuppence (a two-cent coin) and nickels.

• BASIS: 4 cents = tuppence + tuppence.

• INDUCTION: Assume k-cents can be made up from tuppence and nickels \((k = 2t + 5n)\). Show \((k+1)\) can too.
Tuppence and Nickels (cont’d.)

• Case 1: (At least one of each coin: \( k = 2t + 5n \))
  \[
  k + 1 = 2t + 5n + 1 = 2t + 5n + 6 - 5 = (2t + 6) + (5n - 5) = 2(t + 3) + 5(n - 1).
  \]

• Case 2: (No tuppence: \( k = 5n \))
  \[
  k + 1 = 5n + 1 = 5n + 6 - 5 = 2(3) + 5(n - 1).
  \]

• Case 3: (No nickels and \( t > 2 \): \( k = 2t \))
  \[
  k + 1 = 2t + 1 = 2t + 5 - 4 = 2(t - 2) + 5(1).
  \]
Sum of the First $n$ Integers

- Carl Friederich Gauss is bored in school.
- Teach instructs him to add together the numbers from 1 to 100.
- Zip! Carl is done. Sum is 5050. How?

$$1 + 2 + 3 + \ldots + 100$$

$$= (1 + 100) + (2 + 99) + (3 + 98) + \ldots + (50 + 51)$$

$$= 101 + 101 + 101 + \ldots + 101$$

$$= 50(101)$$

$$= 5050.$$
Sum of the First $n$ Integers (cont’d.)

Prove: $$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2} .$$

Proof: (Induction)

Basis: Show: $$\sum_{k=1}^{1} k = \frac{1(2)}{2} .$$

$$\sum_{k=1}^{1} k = 1 \text{ and } \frac{1(2)}{2} = \frac{2}{2} = 1, \text{ so they are equal.}$$
Sum of the First *n* Integers *(cont’d.)*

**Inductive:** Assume for some *p*:

\[ 1 + 2 + \ldots + p = \frac{p(p + 1)}{2}. \]

**Show:** \( (1 + 2 + \ldots + p) + (p + 1) = \frac{(p + 1)(p + 2)}{2}. \)

Now, \( (1 + \ldots + p) + (p + 1) = \frac{p(p + 1)}{2} + (p + 1) \)

\[ = (p + 1)[p/2 + 1] \]

\[ = (p + 1)[p/2 + 2/2] \]

\[ = (p + 1)(p + 2)/2. \]

Therefore \( 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \) for all integers *n*.

**QED**
The Geometric Series

• Any sum of the form: $1 + r + r^2 + r^3 +...+ r^n$ is called a Geometric Series.

• Thus, $1 + 2 + 4 + 8 + 16 +...+ 2^n$ is a geometric series.

• To find the sum of this series, consider:
  
  $$S = 1 + r + r^2 + r^3 +...+ r^n.$$  

  So  
  $$-rS = -r - r^2 - r^3 -...- r^{(n+1)}.$$  

  and  
  $$(1 - r)S = 1 - r^{(n+1)}.$$  

  Therefore,  
  $$1 + r + r^2 +...+ r^n = \frac{1 - r^{(n+1)}}{1 - r}.$$
Proof of the Geometric Series

• Prove: $1 + r + r^2 +\ldots + r^n = \left[r^{(n+1)} - 1\right] / (r - 1)$

• Proof: (Induction) **Basis:** Show true for $n = 0$. LHS = 1. RHS = $[r^{(0+1)} - 1]/(r - 1) = (r-1)/(r-1) = 1$. Therefore LHS = RHS.

**Induction:** Assume $1+r+r^2+\ldots+r^k = r^{(k+1)}-1/r-1$.

Show $1+r+r^2+\ldots+r^k+r^{(k+1)} = \left[r^{(k+2)} - 1\right]/(r - 1)$.

Now, $1+r+r^2+\ldots+r^k+r^{(k+1)} = r^{(k+1)}-1/r-1 + r^{(k+1)}$

$= \left[r^{(k+1)} - 1 + (r - 1)r^{(k+1)} \right]/(r - 1)$

$= \left[r^{(k+1)} - 1 + r \cdot r^{(k+1)} - r^{(k+1)} \right]/(r - 1)$

$= \left[r^{(k+2)} - 1\right]/(r - 1)$. QED