Section 6 - Indirect Argument

• Method of Proof by Contradiction;
• Method of Proof by Contraposition;
• Examples of Each Method.
Proof by Contradiction

• Instead of the Universal Modus Ponens argument form: $\forall x, [P(x) \rightarrow Q(x) \text{ AND } P(x)] \Rightarrow Q(x)$, a Proof by Contradiction (reductio ad absurdum) follows the Universal Modus Tollens form: $\forall x, [P(x) \rightarrow Q(x) \text{ AND } \neg Q(x)] \Rightarrow \neg P(x)$.

• We obtain a contradiction when the conclusion of this form is combined with our standard assumption in a direct proof the $P(x)$ holds.

• This differs marginally from the Method of Contraposition which proves directly the validity of the contrapositive statement.
Method of Proof By Contradiction

• Suppose the statement to be proved is FALSE;
• Show this supposition leads logically to a contradiction (either to the original hypotheses or to some other statement of fact);
• Conclude that the original statement to be proved is TRUE.
Example: No Greatest Integer

**Theorem:** There is no greatest integer.

**Proof:** (Contradiction) Suppose there is a greatest integer $N$. Thus for every integer $k$, $k \leq N$.

Now, since $N$ is an integer, by closure, $(N+1)$ is an integer. Thus: $N + 1 \leq N$, hence $1 \leq 0.*$

Therefore, there is no greatest integer. QED
Sums of Rationals and Irrationals

Theorem: The sum of a rational and an irrational is irrational.

Proof: (Contradiction) Let $r$ be rational, $s$ be irrational, and assume $(r + s)$ is rational. Thus there exist $a, b, c, d \in \mathbb{Z}$, with $r = a/b$, $(r + s) = c/d$ and $b, d \neq 0$.

Now, $s = (r + s) - r = c/d - a/b$

$= (bc - ad)/bd$.

Since $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$, we have $s \in \mathbb{Q}$.*

Therefore $(r + s)$ is irrational. QED
Argument by Contraposition

• Since we know that a statement and its contrapositive are logically equivalent, if we can pose our conjecture in the form of a conditional, we can work, equivalently, with its contrapositive form.

• We call this strategy, simply enough, Argument by Contraposition.
Method of Proof by Contraposition

- Express the statement to be proved in the form \( \forall x, \text{if } P(x) \text{ then } Q(x) \).
- Rewrite this as its contrapositive \( \forall x, \text{if } \neg Q(x) \text{ then } \neg P(x) \).
- Prove the contrapositive form directly:
  - Suppose \( x \) is such that \( Q(x) \) is FALSE.
  - Show that \( P(x) \) is FALSE.
Example of Contraposition

Theorem: Given any integer \( n \), if \( n^2 \) is even, then \( n \) is even.

(Contrapositive: If \( n \) is odd, then \( n^2 \) is odd.)

Proof: (Contraposition) Let \( n \) be an integer and assume that \( n \) is odd. Thus, there is an integer \( k \) such that \( n = 2k + 1 \). Show that \( n^2 \) is odd.

Now, \( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 \)

\[ = 2(2k^2 + 2k) + 1. \]

Since \( k \) is an integer, \( (2k^2 + 2k) \) is an integer.

Therefore \( n^2 \) is odd. QED