Section 2: Reflexivity, Symmetry, and Transitivity

- <u>Definition:</u> Let R be a binary relation on A.
- R is *reflexive* if for all $x \in$ A, $(x,x) \in$ R. (Equivalently, for all x e A, x R x.)
- R is *symmetric* if for all $x,y \in$ A, $(x,y) \in$ R implies $(y,x) \in$ R. (Equivalently, for all $x,y \in$ A, x R y implies that y R x.)
- R is *transitive* if for all $x,y,z \in$ A, $(x,y) \in$ R and $(y,z) \in$ R implies $(x,z) \in$ R. (Equivalently, for all $x,y,z \in$ A, x R y and y R z implies x R z.)

Examples

- Reflexive: The relation R on {1,2,3} given by R = {(1,1), (2,2), (2,3), (3,3)} is reflexive. (*All loops are present.*)
- Symmetric: The relation R on {1,2,3} given by R = {(1,1), (1,2), (2,1), (1,3), (3,1)} is symmetric. (*All paths are 2-way.*)
- Transitive: The relation R on {1,2,3} given by R = {(1,1), (1,2), (2,1), (2,2), (2,3), (1,3)} is transitive. (*If I can get from one point to another in 2 steps, then I can get there in 1 step.*)

Violations of the Properties

- Why is R = {(1,1), (2,2), (3,3)} not reflexive on {1,2,3,4}?

  Because (4,4) is missing.

- Why is R = {(1,2), (2,1), (3,1)} not symmetric?

  Because (1,3) is missing.

- Why is R = {(1,2), (2,3), (1,3), (2,1)} not transitive?

  Because (1,1) and (2,2) are missing.

- Is {(1,1), (2,2), (3,3)} symmetric? transitive?

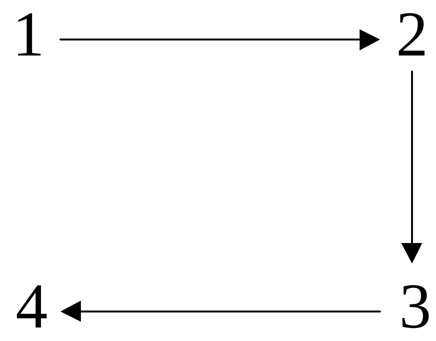
  Yes! Yes!

The Transitive Closure

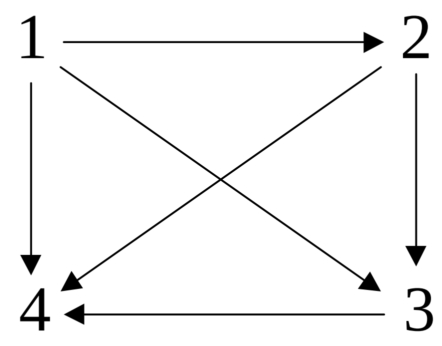
- Definition: Let R be a binary relation on a set A. The *transitive closure* of R is the binary relation R^t on A satisfying the following three properties:
  1. R^t is transitive;
  2. R is a subset of R^t;
  3. If S is any other transitive relation that contains R, then S contains R^t.
- In other words, the transitive closure of R is the *smallest* transitive relation containing R.

Example of the Transitive Closure

- Given the relation R on {1,2,3,4},



its transitive closure is:



Properties of Equality

- Consider the Equality (=) relation on **R**:

  Equality is reflexive since for each $x \in$ **R**, $x = x$.

  Equality is symmetric since for each $x,y \in$ **R**, if $x = y$, then $y = x$.

  Equality is transitive since for each $x,y,z \in$ **R**, if $x = y$ and $y = z$, then $x = z$.
- As a graph, the relation contains only loops, so symmetry and transitivity are vacuously satisfied!

Properties of Congruence Mod p

- Let p be an integer greater than 1, and consider the relation on **Z** given by:
  $$R = \{(x,y) \mid x,y \in \mathbf{Z} \text{ and } x \equiv y \bmod p\}.$$
- When we say $x \equiv y \bmod p$, this means $(x - y) = kp$ for some integer k.
- Now, R is reflexive since $(x - x) = 0 = 0p$, for all integers x.
- Moreover, R is symmetric, since if $x \equiv y \bmod p$, then $(x - y) = kp$, thus $(y - x) = (-k)p$, implying that $y \equiv x \bmod p$.

Congruence Mod p (*cont'd.*)

- Finally, R is transitive. Why?
- Let $x \equiv y \bmod p$ and $y \equiv z \bmod p$. This means there are integers k and j such that $(x - y) = kp$ and $(y - z) = jp$. Hence, $(x - z) = (x - y) + (y - z) = kp + jp = (k + j)p$. Therefore, $x \equiv z \bmod p$.

Properties of Inequality

- Consider the Inequality (< or >) relation on **R**:

  Inequality is *not* reflexive since for no $x \in$ **R** is it true that $x < x$.

  Inequality is *not* symmetric since for each $x,y \in$ **R**, if $x < y$ is true, then $y < x$ is false.

  Inequality is transitive since for each $x,y,z \in$ **R**, if $x < y$ and $y < z$, then $x < z$.

- Inequality is so pathelogically unsymmetric, that we define a special property to describe it.

The Anti-symmetry Property

- Definition: A relation R on a set A is called *anti-symmetric* if $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$.
- This is equivalent to requiring that if $x \neq y$ and $(x,y) \in R$, then $(y,x) \notin R$. (*All streets are one-way*.)
- Example: R = {(1,1), (1,2), (3,2), (3,3)} is anti-symmetric.
- Is every relation symmetric or anti-symmetric?
- No! Consider R = {(1,2), (2,1), (1,3)}.