Chapter 1. Symbolic Logic

- Logical Form and Equivalence
- Conditional Statements
- Valid and Invalid Arguments
- Digital Logic Circuits (Boolean Polynomials)
Logic of Compound Statements

- A *statement* (or *proposition*) is a sentence that is true (T) or false (F), but not both or neither.
- Examples:
  - Today is Monday.
  - $x$ is even and $x > 7$.
  - If $x^2 = 4$, then $x = 2$ or $x = -2$. 
Counterexamples

• If a sentence cannot be judged to be T or F or is not even a sentence, it cannot be a statement.

• Examples:

  Open the door! (*imperative*)
  Did you open the door? (*interrogative*)
  If $x^2 = 4$. (*fragment*)
Compound Statements

• Denote statements using the symbols $p$, $q$, $r$, ...
• Denote the operations $\land$, $\lor$, $\neg$, $\rightarrow$ (to be defined shortly), where:
  
  \[
  p \land q - \text{conjunction of } p \text{ and } q \ (p \text{ and } q);
  \]
  
  \[
  p \lor q - \text{disjunction of } p \text{ and } q \ (p \text{ or } q);
  \]
  
  \[
  \neg p - \text{negation of } p \ (\text{not } p);
  \]
  
  \[
  p \rightarrow q - \text{implication of } p \text{ and } q \ (p \text{ implies } q);
  \]
Compound Statements (cont’d.)

- A Compound statement (or statement form) is a statement which includes at least one operation and one other “atomic” statement.
- For example, “$x = 7$ and $y = 2$” is a compound statement based on the “atomic” statements $p = “x = 7”$ and $q = “y = 2”$.
- In this instance, we can symbolize the compound statement as $r = p \land q$. 
Compound Statements (cont’d.)

- The *Truth Table* of a compound statement is the collection of all the output truth values corresponding to all possible combinations of input truth values of the atomic statements.
- Since each atomic statement can take on 1 of 2 values, 2 inputs have 4 combinations, 3 inputs have 8, 4 inputs have 16, 5 inputs have 32, etc.
Logical Operations

• Negation: \[ p \quad \sim p \]
  \[
  \begin{array}{cc}
  T & F \\
  F & T \\
  \end{array}
  \]

• Conjunction: \[ p \quad q \quad (p \land q) \]
  \[
  \begin{array}{ccc}
  T & T & T \\
  T & F & F \\
  F & T & F \\
  F & F & F \\
  \end{array}
  \]

• Disjunction: \[ p \quad q \quad (p \lor q) \]
  \[
  \begin{array}{ccc}
  T & T & T \\
  T & F & T \\
  F & T & T \\
  F & F & F \\
  \end{array}
  \]
Example: \((p \lor q) \land \sim r\)

- Proceed from left to right:

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<th>((p \lor q) \land \sim r)</th>
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1.1.8
Logical Equivalence

• Two *compound statements* are logically equivalent if they have the same truth table. We denote this as $p \equiv q$.

• \[
\begin{array}{c|c|c|c}
  p & \sim p & \sim(\sim p) \\
  \hline
  T & F & T \\
  F & T & F \\
\end{array}
\]
  hence $p \equiv \sim(\sim p)$.

• $\sim(p \land q) \equiv \sim p \land \sim q$ ?
  No, since $\sim(T \land F) \equiv T$, but $(\sim T \land \sim F) \equiv F$. 

Tautology & Contradiction

- A statement whose truth table is all “T” is called a tautology, denoted as $p \equiv t$.
- A statement whose truth table is all “F” is called a contradiction, denoted as $p \equiv c$.
- Clearly, $\neg t \equiv c$ and $\neg c \equiv t$.
- Are all logical statements either tautology or contradiction?
Algebra of Symbolic Logic

- **Commutative Laws:**
  \[ p \land q \equiv q \land p \]
  \[ p \lor q \equiv q \lor p \]

- **Associative Laws:**
  \[ (p \land q) \land r \equiv p \land (q \land r) \]
  \[ (p \lor q) \lor r \equiv p \lor (q \lor r) \]

- **Distributive Laws:**
  \[ p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \]
  \[ p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \]
Algebra of Symbolic Logic

• Identity Laws:

\[ p \land t \equiv p \]
\[ p \lor c \equiv p \]

• Negation Laws:

\[ p \land \neg p \equiv c \]
\[ p \lor \neg p \equiv t \]

• Double Negative Laws: \( \neg(\neg p) \equiv p \)

• Negations of \( t \) and \( c \):

\[ \neg t \equiv c \]
\[ \neg c \equiv t \]
Algebra of Symbolic Logic

• Idempotent Laws: \( p \land p \equiv p \)  \( p \lor p \equiv p \)

• DeMorgan’s Laws:
  \[ \neg(p \land q) \equiv \neg p \lor \neg q \]
  \[ \neg(p \lor q) \equiv \neg p \land \neg q \]

• Universal Bound Laws: \( p \land c \equiv c \)  \( p \lor t \equiv t \)

• Absorption Laws:
  \[ p \land (p \lor q) \equiv p \]
  \[ p \lor (p \land q) \equiv p \]