Basic Trigonometry

• Angles
  - degrees = $180/\pi$ radians

• Some functions
  - $\sin \theta = o/h$
  - $\cos \theta = a/h$
  - $\tan \theta = o/a$

• A few identities
  - $\sin(-A) = -\sin A$
  - $\cos(-A) = \cos A$
  - $\sin^2 A + \cos^2 A = 1$
Vectors

- Vector: length and direction gives offset

\[ \mathbf{a} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \quad \mathbf{a}^T = \begin{bmatrix} x_a & y_a & z_a \end{bmatrix} \]

Vectors (2)

- Operations
  - Length
    \[ \| \mathbf{a} \| = \sqrt{x_a^2 + y_a^2 + z_a^2} \]
  - Addition
    \[ \mathbf{a} + \mathbf{b} = \begin{bmatrix} x_a + x_b \\ y_a + y_b \\ z_a + z_b \end{bmatrix} \]
  - Subtraction
    \[ \mathbf{a} - \mathbf{b} = \begin{bmatrix} x_a - x_b \\ y_a - y_b \\ z_a - z_b \end{bmatrix} \]
Vectors (3)

- Dot Product
  \[ \mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b \]
  - angle between
  \[ \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \]
  - projection of one on other

\[ \mathbf{a} \rightarrow \mathbf{b} = \|\mathbf{a}\| \cos \theta \]

Vectors (4)

- Cross Product
  \[ \mathbf{a} \times \mathbf{b} = (y_a z_b - z_a y_b, z_a x_b - x_a z_b, x_a y_b - y_a x_b) \]

- \( \mathbf{c} = \mathbf{a} \times \mathbf{b} \)
  - orthogonal to \( \mathbf{a}, \mathbf{b} \)
  - length=pgrom area
- \( \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} \)
Vectors (5)

- Coordinate Frames (uvw coord system)
  - Orthonormal basis
    - Unit length \( \|u\| = \|v\| = \|w\| = 1 \)
    - Orthogonal \( u \cdot v = v \cdot w = w \cdot u = 0 \)
  - Right-handed vs left-handed
    - Right-handed \( W = u \times V \)

- Coordinate frames
  - Global (world) coordinate system
  - Local (object) coordinate system

Orthonormal Basis

- Constructing from a vector \( a \)
  - Unit vector in direction of \( a \): \( w = \frac{a}{\|a\|} \)
  - Any perpendicular to \( w \) (using noncollinear \( t \)): \( u = \frac{t \times w}{\|t \times w\|} \)
  - Unit vector perpendicular to both: \( v = w \times u \)

- Constructing from two vectors \( a, b \)
  - Unit vector in direction of \( a \): \( w = \frac{a}{\|a\|} \)
  - Perpendicular to \( w \) and \( b \): \( u = \frac{b \times w}{\|b \times w\|} \)
  - Unit vector perpendicular to both: \( v = w \times u \)
Linear Interpolation

- Formula
  \[ n = n_1 + t(n_2 - n_1) \]
  where \( n = n_1 \) at \( t=0 \), \( n = n_2 \) at \( t=1 \)

- Uses
  - Points along a line (repeat for \( x,y,z \))
  - Colors between sample points (repeat for \( r,g,b \))

Implicit Lines

- Function defines relationship among coords
- 2D Formula
  \[ y - mx - b = 0 \]
  - where \( y \) is slope and \( b \) is intercept
- General form
  \[ Ax + By + C = 0 \]
  - Orthogonal to \( [A \ B] \)
  - Computing from points
    \[ (y_0 - y_1)x + (x_0 - x_1)y + x_0y_1 - x_1y_0 = 0 \]
    - Substitute in point for signed distance
Implicit Primitives

- Function defines relationship among coords
- Implicit Plane
  \[(p - a) \cdot n = 0\]
  - Orthogonal to \(n\), through \(a\)
  - Computing from points \(a, b, c\)
  \[(p - a) \cdot ((b - a) \times (c - a)) = 0\]
- Implicit Sphere
  \[(p - c)^2 - r^2 = 0\]
  - Radius \(r\), center point \(c\)

Parametric Lines

- Define line by end points
- Formula
  \[p = p_1 + t(p_2 - p_1)\]
  where \(p = p_1\) at \(t=0\), \(p = p_2\) at \(t=1\)
- Components
  \[x = x_1 + t(x_2 - x_1)\]
  \[y = y_1 + t(y_2 - y_1)\]
  \[z = z_1 + t(z_2 - z_1)\]
**Triangles**

- Specified by triplet of vertex positions
  - a,b,c
  - Counter-clockwise order

- Finding triangle normal
  \[ n = (b - a) \times (c - a) \]

**Barycentric Coordinates**

- Use non-orthogonal coordinates to describe position relative to vertices
  \[ p = a + \beta(b - a) + \gamma(c - a) \]
  \[ p(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c \]
  - Coordinates correspond to scaled signed distance from lines through pairs of vertices
Barycentric Example

Barycentric Coordinates

- Computing coordinates

\[ \gamma = \frac{(y_a - y_b)x + (x_b - x_a)y + x_ay_b - x_by_a}{(y_a - y_b)x_c + (x_b - x_a)y_c + x_ay_b - x_by_a} \]

\[ \beta = \frac{(y_a - y_c)x + (x_c - x_a)y + x_ay_c - x_cy_a}{(y_a - y_c)x_b + (x_c - x_a)y_b + x_ay_c - x_cy_a} \]

\[ \alpha = 1 - \beta - \gamma \]
Matrix Multiplication

• With matrices A, B

\[
A = \begin{bmatrix}
a_{00} & a_{10} & a_{20} \\
a_{10} & a_{11} & a_{12} \\
a_{20} & a_{21} & a_{22} \\
\end{bmatrix} \quad B = \begin{bmatrix}
b_{00} & b_{10} & b_{20} \\
b_{10} & b_{11} & b_{12} \\
b_{20} & b_{21} & b_{22} \\
\end{bmatrix}
\]

• To compute C=AB

\[
c_{ij} = a_{i0}b_{0j} + a_{i1}b_{1j} + a_{i2}b_{2j}
\]