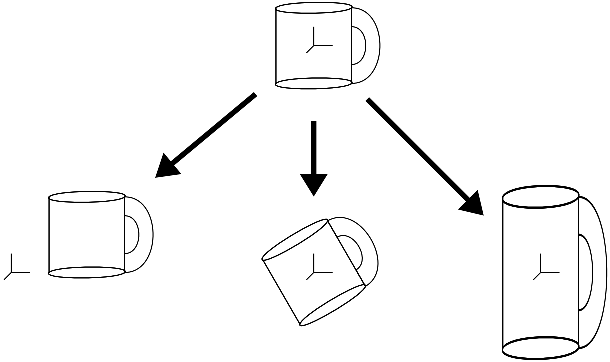
3D Transformations

CMSC 435/634

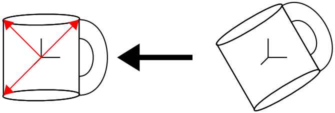
Transformation

Webster: The operation of changing one configuration or expression into another in accordance with a mathematical rule



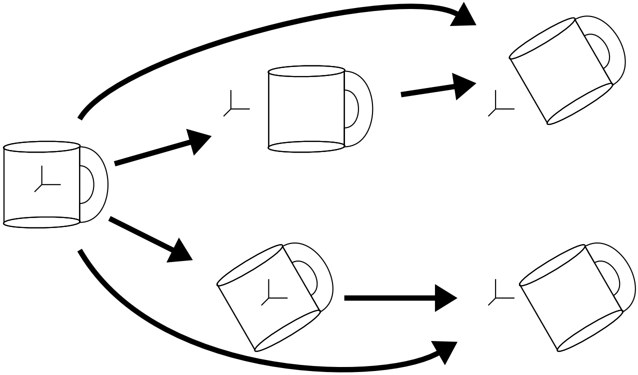
Using Transformation

- Points on object represented as vector offset from origin
- Transform is a vector to vector function
  - $\vec{p'} = f(\vec{p})$
- Relativity:
  - From $\vec{p'}$ point of view, object is transformed
  - From $\vec{p}$ point of view, coordinate system changes
- Inverse transform, $\vec{p} = f^{-1}(\vec{p'})$



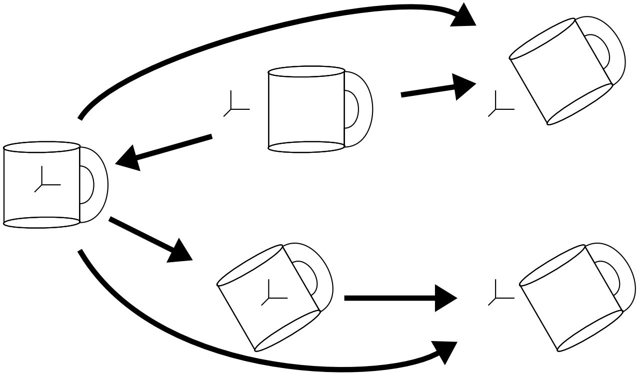
Composing Transforms

- Order matters
    - $R(T(\vec{p})) = R \circ T(\vec{p})$
    - $T(R(\vec{p})) = T \circ R(\vec{p})$



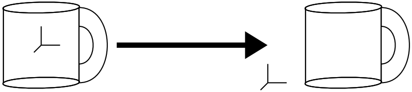
Inverting Composed Transforms

- Reverse order
  - $(R \circ T)^{-1}(\vec{p'}) = T^{-1}(R^{-1}(\vec{p'}))$
  - $(T \circ R)^{-1}(\vec{p'}) = R^{-1}(T^{-1}(\vec{p'}))$



Translation

- $\vec{p'} = \vec{p} + \vec{t}$
- $\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \end{bmatrix}$
- $\vec{t}$ says where $\vec{p}$-space origin ends up ($\vec{p'} = \vec{0} + \vec{t}$)
- Composition: $\vec{p'} = (\vec{p} + \vec{t_0}) + \vec{t_1} = \vec{p} + (\vec{t_0} + \vec{t_1})$

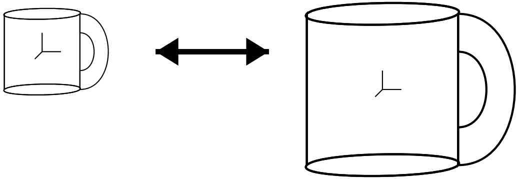


Linear Transforms

- $\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$
- Matrix says where $\vec{p}$-space axes end up
  - $\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
    $\begin{bmatrix} b \\ e \\ h \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
    $\begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
  - Composition: $\vec{p'} = M\ (N\ \vec{p}) = (M\ N)\vec{p}$

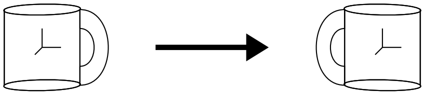
Common case: Scaling

- $\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} s_x \ p_x \\ s_y \ p_y \\ s_z \ p_z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$
- Inverse: $\begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1/s_z \end{bmatrix}$

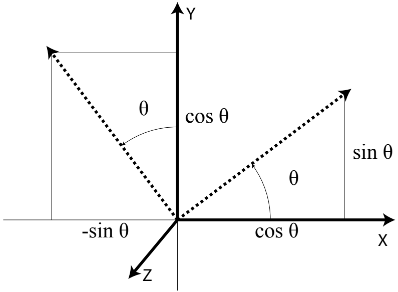


Common case: Reflection

- Negative scaling
- $\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} -p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$

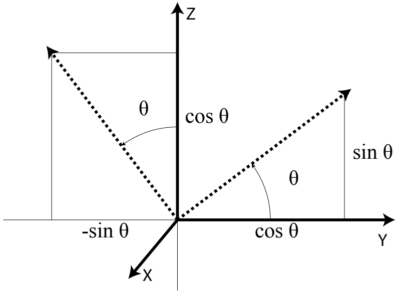


Common case: Rotation



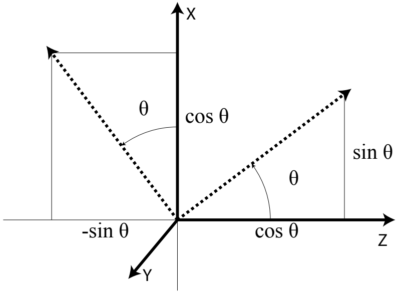
- Orthogonal, so $M^{-1} = M^T$
- Rotate around Z: $\vec{p'} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{p}$

Common case: Rotation



- Orthogonal, so $M^{-1} = M^T$
- Rotate around X: $\vec{p'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \vec{p}$

Common case: Rotation



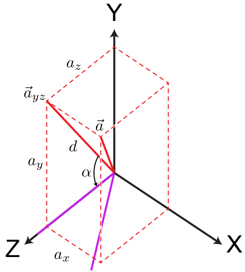
- Orthogonal, so $M^{-1} = M^T$
- Rotate around Y: $\vec{p'} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \vec{p}$

Composing Transforms

- Scale by s along axis $\vec{a}$
  - Rotate to align $\vec{a}$ with Z
  - Scale along Z
  - Rotate back

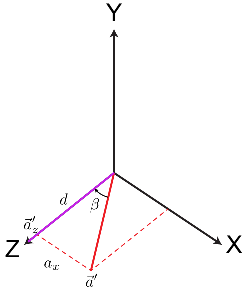
Rotate by α around X into XZ plane

- Projection of $\vec{a}$ onto YZ: $\vec{a}_{yz} = \begin{bmatrix} 0 \\ a_y \\ a_z \end{bmatrix}$
- length $d = \sqrt{(a_y)^2 + (a_z)^2}$
- So $\cos\alpha = a_z/d$, $\sin\alpha = a_y/d$
- $R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_z/d & -a_y/d \\ 0 & a_y/d & a_z/d \end{bmatrix}$
- Result $\vec{a'} = \begin{bmatrix} a_x \\ 0 \\ d \end{bmatrix}$



Rotate by -β around Y to Z axis

- $\vec{a'} = \begin{bmatrix} a_x \\ 0 \\ d \end{bmatrix}$
- length $= 1$
- So $\cos\beta = d$, $\sin\beta = a_x$
- $R_Y = \begin{bmatrix} d & 0 & -a_x \\ 0 & 1 & 0 \\ a_x & 0 & d \end{bmatrix}$
- Result $\vec{a'}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



Composing Transforms

- Scale by s along Z: $S_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$
- Scale by s along axis $\vec{a}$
  - Rotate to align $\vec{a}$ with Z
  - Scale along Z
  - Rotate back
  - $\vec{p'} = R_X^{-1} R_Y^{-1} S_Z R_Y R_X \vec{p}$

Affine Transforms

- Affine = Linear + Translation
- Composition? $A\ (B\ \vec{p} + \vec{t_0}) + \vec{t_1} = A\ B\ \vec{p} + A\ \vec{t_0} + \vec{t_1}$
- Yuck!

Homogeneous Coordinates

- Add a '1' to each point
- $\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$
- $\vec{p'}_x = (a\ p_x + b\ p_y + c\ p_z) + t_x$
- $\vec{p'}_y = (d\ p_x + e\ p_y + f\ p_z) + t_y$
- $\vec{p'}_z = (g\ p_x + h\ p_y + i\ p_z) + t_z$
- $1 = (0p_x + 0p_y + 0p_z) + 1$

Homogeneous Coordinates

- $$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$
- $\vec{p}' = \left[\begin{array}{ccc|c} \vec{x} & \vec{y} & \vec{z} & \vec{t} \end{array}\right] \vec{p}$
  - $\vec{t}$ says where the $\vec{p}$-space origin ends up
  - $\vec{x}$, $\vec{y}$, $\vec{z}$ say where the $\vec{p}$-space axes end up
- Composition: Just matrix multiplies!

Composing Transforms

- Rotate by θ about line between $\vec{p}_0$ and $\vec{p}_1$:
  - Translate $\vec{p}_0$ to origin
  - Rotate to align $\vec{p}_1 - \vec{p}_0$ with Z
  - Rotate by θ around Z
  - Undo $\vec{p}_1 - \vec{p}_0$ rotation
  - Undo translation
- $T^{-1} R_X^{-1} R_Y^{-1} R_Z(\theta) R_Y R_X T$

Vectors

- Transform by *Jacobian Matrix*
- Matrix of partial derivatives
  - $\begin{bmatrix} \vec{p'}_x \\ \vec{p'}_y \\ \vec{p'}_z \end{bmatrix} = \begin{bmatrix} a\ p_x + b\ p_y + c\ p_z + t_x \\ d\ p_x + e\ p_y + f\ p_z + t_y \\ g\ p_x + h\ p_y + i\ p_z + t_z \end{bmatrix}$
  - $J = \begin{bmatrix} \partial p'_x/\partial p_x & \partial p'_x/\partial p_y & \partial p'_x/\partial p_z \\ \partial p'_y/\partial p_x & \partial p'_y/\partial p_y & \partial p'_y/\partial p_z \\ \partial p'_z/\partial p_x & \partial p'_z/\partial p_y & \partial p'_z/\partial p_z \end{bmatrix}$
  - $J = \begin{bmatrix} a & b & c \\ c & d & f \\ g & h & i \end{bmatrix}$
- *Upper-left 3x3*

Normals

- Normal should remain perpendicular to tangent vector
- $\vec{n} \cdot \vec{v} = \vec{n'} \cdot \vec{v'} = 0$
- $\begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \left(\begin{bmatrix} n_x & n_y & n_z \end{bmatrix} J^{-1} \right) \left(J \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \right) = 0$
- $\vec{n'} = \vec{n} J^{-1}$
- Multiply by inverse on right
- OR multiply *column* normal by inverse transpose
  - $(J^{-1})^T = J$ if J is orthogonal (only rotations)