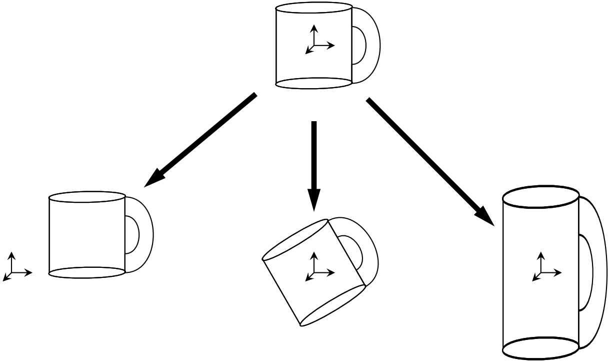
3D Transformations

CMSC 435/634

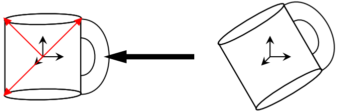
Transformation

Webster: The operation of changing one configuration or expression into another in accordance with a mathematical rule



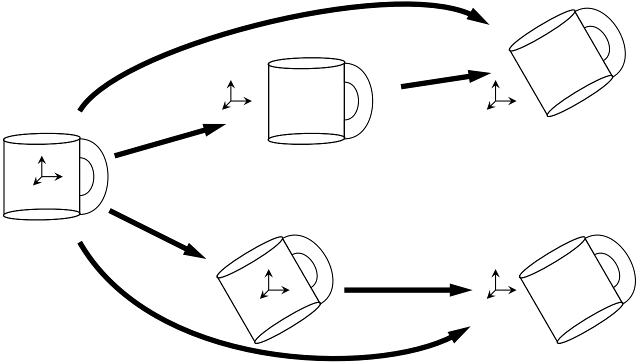
Using Transformation

- Points on object represented as vector offset from origin
- Transform is a vector to vector function
  - $\vec{p'} = f(\vec{p})$
- Relativity:
  - From $\vec{p'}$ point of view, object is transformed
  - From $\vec{p}$ point of view, coordinate system changes
- Inverse transform, $\vec{p} = f^{-1}(\vec{p'})$



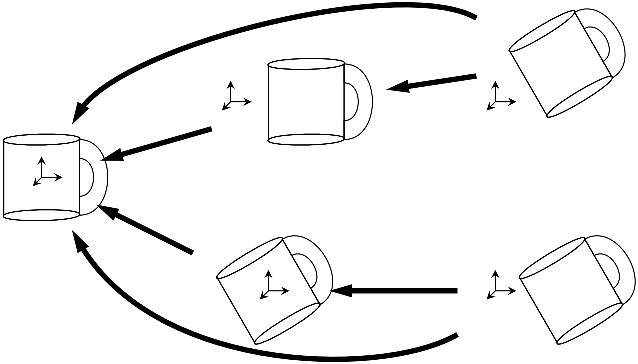
Composing Transforms

- Order matters
  - $R(T(\vec{p})) = R \circ T(\vec{p})$
  - $T(R(\vec{p})) = T \circ R(\vec{p})$



Inverting Composed Transforms

- Reverse order
  - $(R \circ T)^{-1}(\vec{p'}) = T^{-1}(R^{-1}(\vec{p'}))$
  - $(T \circ R)^{-1}(\vec{p'}) = R^{-1}(T^{-1}(\vec{p'}))$



Translation

- $\vec{q} = \vec{p} + \vec{t}$
- $\vec{t}$ says where $\vec{p}$-space origin ends up in $\vec{q}$-space
  - $\vec{q} = \vec{0} + \vec{t}$
- Composition:
  - $$\begin{aligned} \vec{q} &= (\vec{p} + \vec{t_0}) + \vec{t_1} \\ &= \vec{p} + (\vec{t_0} + \vec{t_1}) \end{aligned}$$

Linear Transforms

- $$\begin{bmatrix} q^x \\ q^y \\ q^z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$$
- Matrix says where $\vec{p}$-space axes end up in $\vec{q}$-space
  - $$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
    $$\begin{bmatrix} b \\ e \\ h \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
    $$\begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
  - Composition:
    - $$\begin{aligned} \vec{q} &= M\,(N\,\vec{p}) \\ &= (M\,N)\vec{p} \end{aligned}$$

Scaling

- $$\begin{bmatrix} s_x p^x \\ s_y p^y \\ s_z p^z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$$
- Inverse: $\begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1/s_z \end{bmatrix}$

Reflection

- Negative scaling
- $$\begin{bmatrix} -p^x \\ p^y \\ p^z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$$

Rotate

- Rotate around X: $\vec{q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \vec{p}$
- Rotate around Y: $\vec{q} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \vec{p}$
- Rotate around Z: $\vec{q} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{p}$
- Orthogonal, so $M^{-1} = M^T$

Affine Transforms

- Affine = Linear + Translation
- Composition?
  - $A\ (B\ \vec{p} + \vec{t_0}) + \vec{t_1} = A\ B\ \vec{p} + A\ \vec{t_0} + \vec{t_1}$
- Yuck!

Homogeneous Coordinates

- Add a '1' to each point
- $$\begin{bmatrix} q^x \\ q^y \\ q^z \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} a & b & c & t^x \\ d & e & f & t^y \\ g & h & i & t^z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} p^x \\ p^y \\ p^z \\ 1 \end{bmatrix}$$
- $\vec{q} = \left[\begin{array}{ccc|c} \vec{x} & \vec{y} & \vec{z} & \vec{t} \end{array}\right] \vec{p}$
  - $\vec{t}$ says where the $\vec{p}$-space origin ends up
  - $\vec{x}$, $\vec{y}$, $\vec{z}$ say where the $\vec{p}$-space axes end up
- Composition: Just matrix multiplies!