Lines, Planes and Triangles

CMSC 435/634

Implicit Lines and Planes

Lines / 2D

$$a\,X + b\,Y + d = 0$$

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = -d$$

Planes / 3D

$$a\,X + b\,Y + c\,Z + d = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = -d$$

$$\begin{aligned} \vec{n} \bullet \vec{P} &= -d \\ &= \vec{n} \bullet \vec{P}_0 \end{aligned}$$

Homogeneous Equations

- All terms of the same degree

Lines / 2D

$$a\ X + b\ Y + d = 0$$

Planes / 3D

$$a\ X + b\ Y + c\ Z + d = 0$$

Multiply through by w

$$a\ X\ \ w + b\ Y\ \ w + d\ \ w = 0$$

$$a\ x + b\ y + d\ \ w = 0$$

$$\begin{bmatrix} a & b & d \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$a\ X\ \ w + b\ Y\ \ w + c\ Z\ \ w + d\ \ w = 0$$

$$a\ x + b\ y + c\ z + d\ \ w = 0$$

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$$\left[\ \vec{n}\ \middle|\ -\vec{n} \bullet \vec{P}_0\ \right] \vec{p} = 0$$

Parametric Lines and Planes

Lines

$\vec{p} = \vec{p}_0 + t\ \vec{v}$

Planes

$\vec{p} = \vec{p}_0 + s\ \vec{u} + t\ \vec{v}$

Tangents

$\frac{d\vec{p}}{dt} = \vec{v}$

$\frac{\partial \vec{p}}{\partial s} = \vec{u}; \frac{\partial \vec{p}}{\partial t} = \vec{v}$

Normals

$\vec{n} = \hat{z} \times \vec{v} = \begin{bmatrix} -v^y & v^x \end{bmatrix}$

$\vec{n} = \vec{u} \times \vec{v}$

Given Points on Line or Plane

Lines

$\vec{p} = \vec{p}_0 + t\ \vec{v}$

$\vec{p} = \vec{p}_0 + t\ (\vec{p}_1 - \vec{p}_0)$

$\vec{p}(0) = \vec{p}_0$

$\vec{p}(1) = \vec{p}_0 + \vec{p}_1 - \vec{p}_0 = \vec{p}_1$

Planes

$\vec{p} = \vec{p}_0 + s\ \vec{u} + t\ \vec{v}$

$\vec{p} = \vec{p}_0 + s\ (\vec{p}_1 - \vec{p}_0) + t\ (\vec{p}_2 - \vec{p}_0)$

$\vec{p}(0,0) = \vec{p}_0$

$\vec{p}(1,0) = \vec{p}_0 + \vec{p}_1 - \vec{p}_0 = \vec{p}_1$

$\vec{p}(0,1) = \vec{p}_0 + \vec{p}_2 - \vec{p}_0 = \vec{p}_2$

Barycentric Form

Lines	**Planes**
$\vec{p} = \vec{p}_0 + t\ (\vec{p}_1 - \vec{p}_0)$	$\vec{p} = \vec{p}_0 + s\ (\vec{p}_1 - \vec{p}_0) + t\ (\vec{p}_2 - \vec{p}_0)$

Rearrange for weighted sum of points

$\vec{p} = (1 - t)\ \vec{p}_0 + t\ \vec{p}_1$	$\vec{p} = (1 - s - t)\ \vec{p}_0 + s\ \vec{p}_1 + t\ \vec{p}_2$
$\vec{p} = r\ \vec{p}_0 + t\ \vec{p}_1$	$\vec{p} = r\ \vec{p}_0 + s\ \vec{p}_1 + t\ \vec{p}_2$
where $r + t = 1$	where $r + s + t = 1$

- r, s and t are the barycentric coordinates of $\vec{p}$

Computing Barycentric Coordinates: System of Equations

- r, s, t as linear equations
  - $r = \begin{bmatrix} a & b & d \end{bmatrix} \vec{p}$
  - Three unknowns, a, b and d
  - Three constraints, $r = 1$ @ $\vec{p}_0$, $r = 0$ @ $\vec{p}_1$, $r = 0$ @ $\vec{p}_2$
    - $1 = \begin{bmatrix} a & b & d \end{bmatrix} \vec{p}_0$
    - $0 = \begin{bmatrix} a & b & d \end{bmatrix} \vec{p}_1$
    - $0 = \begin{bmatrix} a & b & d \end{bmatrix} \vec{p}_2$
  - $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b & d \end{bmatrix} \begin{bmatrix} \vec{p}_0 & \vec{p}_1 & \vec{p}_2 \end{bmatrix}$
  - $\begin{bmatrix} a & b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{p}_0 & \vec{p}_1 & \vec{p}_2 \end{bmatrix}^{-1}$

Computing Barycentric Coordinates: Ratio of Heights

- r is 0 at $\vec{p}_1$, $\vec{p}_2$, and all points on the $\overline{\vec{p}_1\vec{p}_2}$ line
- r is 1 at $\vec{p}_0$
- r measures the perpendicular height of $\vec{p}$ above $\overline{\vec{p}_1\vec{p}_2}$
- Can measure with dot product against normal to $\overline{\vec{p}_1\vec{p}_2}$
  - Apply Gram-Schmidt orthogonalization
  - $\vec{e}_{1,2} = \vec{p}_2 - \vec{p}_1$
  - $\vec{e}_{1,0} = \vec{p}_0 - \vec{p}_1$
  - $\vec{n}_r = \vec{e}_{1,0} - \frac{\vec{e}_{1,0} \bullet \vec{e}_{1,2}}{\vec{e}_{1,2} \bullet \vec{e}_{1,2}} \vec{e}_{1,2}$
- Measure height of triangle: $h = \vec{n}_r \bullet \vec{p}_0 - \vec{n}_r \bullet \vec{p}_1$
- Measure height of point $\vec{p}$: $h_p = \vec{n}_r \bullet \vec{p} - \vec{n}_r \bullet \vec{p}_1$
- r is the ratio h_p/h

Computing Barycentric Coordinates: Ratio of Areas

- Triangle area $= \frac{1}{2} width\ height$
- $\therefore$ Ratio of heights = ratio of triangle area (with same base)
  - $\frac{1}{2}$ and *width* terms cancel
- $area(\vec{p}_0, \vec{p}_1, \vec{p}_2) = \frac{1}{2} w\ h$
- $area(\vec{p}, \vec{p}_1, \vec{p}_2) = \frac{1}{2} w\ h_p$
- $r = \frac{area(\vec{p},\vec{p}_1,\vec{p}_2)}{area(\vec{p}_0,\vec{p}_1,\vec{p}_2)} = \frac{\frac{1}{2} w\ h_p}{\frac{1}{2} w\ h} = h_p/h$

Computing Barycentric Coordinates: Cross Product

- Magnitude of cross product is twice area of triangle
- ∴ Ratio of areas = ratio of cross products
  - $\vec{n}_p = (\vec{p}_2 - \vec{p}_1) \times (\vec{p} - \vec{p}_1)$
  - $\vec{n}_{p_0} = (\vec{p}_2 - \vec{p}_1) \times (\vec{p}_0 - \vec{p}_1)$
  - $|r| = |\vec{n}_p|/|\vec{n}_{p_0}| = |w\ h_p|/|w\ h| = |h_p|/|h|$
- Sign: positive if $\vec{n}_p$ and $\vec{n}_{p_0}$ point the same direction
  - r is positive if $\vec{n}_p \bullet\ \vec{n}_{p_0} > 0$
  - r is negative if $\vec{n}_p \bullet\ \vec{n}_{p_0} < 0$
- For triangle in 2D; x,y components of cross product are 0
  - $r = \vec{n}_p^z / \vec{n}_{p_0}^z$

Using Barycentric Coordinates: Point in Triangle Test

- Point $\vec{p}$ is in triangle $\triangle\, \vec{p}_0\vec{p}_1\vec{p}_2$
  - iff $r \geq 0, s \geq 0, t \geq 0$
- Each barycentric coordinate is one edge test
  - $r > 0$ on the inside of $\overline{\vec{p}_1\vec{p}_2}$
  - $s > 0$ on the inside of $\overline{\vec{p}_2\vec{p}_0}$
  - $t > 0$ on the inside of $\overline{\vec{p}_0\vec{p}_1}$
- Optimizations
  - Only need sign, can avoid division
  - For known vertical or horizontal edges, reduces to $\vec{p}^x - \vec{p}_0^x \geq 0$
- For grid (as in assignment), can locate grid triangle without barycentric coordinates
  - $i = floor(x/spacing); j = floor(y/spacing)$
  - Single dot product/edge test determines top vs. bottom triangle in cell

Using Barycentric Coordinates: Interpolation

- Given r, s and t, can interpolate position, $\vec{p}$ within the triangle
  - $\vec{p} = r\ \vec{p}_0 + s\ \vec{p}_1 + t\ \vec{p}_2$
- Given $\vec{p}$ can compute r, s and t.
- Use these coordinates to interpolate other per-vertex values
  - $z = r\ z_0 + s\ z_1 + t\ z_2$
  - $color = r\ color_0 + s\ color_1 + t\ color_2$
  - $\vec{n} = r\ \vec{n}_0 + s\ \vec{n}_1 + t\ \vec{n}_2$