CMSC 441: Homework #2 Solutions

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Exercise 4.1–1

Show that the solution of $T(n) = T(\lceil n/2 \rceil) + 1$ is $O(lg\ n)$

Solution:
We guess that the solution is $T(n) = O(lg\ n)$. We must prove that $T(n) \leq c\,lg\ n$ for appropriate $c > 0$. We start by assuming that this bound holds for $\lceil n/2 \rceil$, that is that $T(\lceil n/2 \rceil) \leq c\,lg(\lceil n/2 \rceil)$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\leq c\,lg(\lceil n/2 \rceil) + 1 \\ &\leq c\,lg(n/2) + 1 \\ &= c\,lg\,n - c\,lg 2 + 1 \\ &= c\,lg\,n - c + 1 \\ &= c\,lg\,n && \text{if } c \geq 1 \end{aligned}$$

the last step holds as long as $c \geq 1$. For the base case, it suffices* to show that $T(2) \leq c\,lg\,2$ for some $c \geq 1$. $T(2) = T(1) + 1$ or $T(2) = 2$ assuming $T(1) = 1$. Thus $T(2) \leq c\,lg\,2$ if $c \geq 2$.

Exercise 4.1–2

We saw that the solution of $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $O(n\ lg\,n)$. Show that the solution of this recurrence is also $\Omega(n\,lg\ n)$. Conclude that the solution is $\Theta(n\ lg\,n)$.

Solution:
We guess that the solution is $T(n) = \Omega(n\,lg\ n)$. We must prove that $T(n) \geq cn\,lg\ n$ for appropriate $c > 0$. We start by assuming that this bound holds for $\lfloor n/2 \rfloor$, that is that $T(\lfloor n/2 \rfloor) \geq c\lfloor n/2 \rfloor lg(\lfloor n/2 \rfloor)$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\geq 2c\lfloor n/2 \rfloor\,lg(\lfloor n/2 \rfloor) + n \\ &\geq cn\,lg(n/2) + n \\ &= cn\,lg\,n - cn\,lg 2 + n \\ &= c\,lg\,n - cn + n \\ &= cn\,lg\,n && \text{if } c \leq 1 \end{aligned}$$

the last step holds as long as $c \geq 1$. For the base case, we must show that $T(2) \leq c\,lg\,2$ and $T(3) \leq c\,lg\,3^{\dagger}$ for some $c \geq 1$. $T(2) = 2T(1) + 2$ or $T(2) = 4$ assuming $T(1) = 1$. Similarly, $T(3) = 2T(1) + 3$ or $T(2) = 5$.Thus $T(2) \geq c\,lg\,2$ and $T(3) \geq c\,lg\,3$ if $c \leq 1$.

Exercise 4.1–6

Solve the recurrence $T(n) = 2T(\sqrt{n}) + 1$ by making a change of variables. Your solution should be asymptotically tight. Do not worry about whether values are integral.

*In this exercise, it is not necessary to show $T(3) \leq c\,lg\,3$. Why ?

†In this exercise, it *is* necessary to show $T(3) \leq c\,lg\,3$ as well. Why ?

Solution:

Let $m = lg\,n$, thus,

$$T(2^m) = 2T(2^{m/2}) + 1,$$

We now rename $S(m) = T(2^m)$ to produce new recurrence

$$S(m) = 2S(m/2) + 1.$$

From the arguments on similar lines to first exercise, $S(m) = \Theta(m)$. Changing back to $T(n)$ and re–substituting $m = lg\,n$, $T(n) = \Theta(lg\,n)$.