CMSC 441: Homework #1 Solutions

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Exercise 3.1–2
Show that for any real constants \( a \) and \( b \), where \( b > 0 \),
\[(n + a)^b = \Theta(n^b)\]

Solution:

\[
(n + a)^b \leq (n + |a|)^b, \text{ where } n > 0
\]
\[
\leq (n + n)^b \text{ for } n \geq |a|
\]
\[
= (2n)^b
\]
\[
= c_1 \cdot n^b, \text{ where } c_1 = 2^b
\]

Thus

\[
(n + a)^b = \Omega(n^b). \tag{1}
\]

\[
(n + a)^b \geq (n - |a|)^b, \text{ where } n > 0
\]
\[
\geq (c'_2 n)^b \text{ for } c'_2 = 1/2 \text{ where } n \geq 2|a|
\]
\[
as \frac{n}{2} \leq n - |a|, \text{ for } n \geq 2|a|
\]

Thus

\[
(n + a)^b = O(n^b) \tag{2}
\]

The result follows from 1 and 2 with \( c_1 = 2^b, c_2 = 2^{-b} \), and \( n_0 \geq 2|a| \).

Exercise 3.1–4
Is \( 2^{n+1} = O(2^n) \)? Is \( 2^{2n} = O(2^n) \)?

Solution:

(a)
Is \( 2^{n+1} = O(2^n) \)? Yes.

\[
2^{n+1} = 2 \cdot 2^n \leq c \cdot 2^n \text{ where } c \geq 2.
\]

(b)
Is \( 2^{2n} = O(2^n) \)? No.

\[
2^{2n} = 2^n \cdot 2^n. \text{ Suppose } 2^{2n} = O(2^n). \text{ Then there is a constant } c > 0 \text{ such that } c > 2^n. \text{ Since } 2^n \text{ is unbounded, no such } c \text{ can exist.}
\]
Exercise 3.1–7

Prove that \( o(g(n)) \cap \omega(g(n)) \) is the empty set.

Solution:
Suppose not. Let \( f(n) \in o(g(n)) \cap \omega(g(n)) \) Now \( f(n) = \omega(g(n)) \) if and only if \( g(n) = o(f(n)) \) and \( f(n) = o(g(n)) \) by assumption. By transitivity property (page 49), \( f(n) = o(f(n)) \) i.e. for all constants \( c > 0, f(n) < cf(n) \). Choose \( c < 1 \) and we have the desired contradiction from the asymptotic nonnegativity of \( f(n) \).

Exercise 3.1–8

We can extend our notation to the case of two parameters \( n \) and \( m \) that can go to infinity independently at different rates. For a given function \( g(n, m) \), we denote by \( O(g(n, m)) \) the set of functions

\[
O(g(n, m)) = \{ f(n, m) : \text{there exist positive constants } c, n_0, \text{ and } m_0 \text{ such that } 0 \leq f(n, m) \leq cg(n, m) \text{ for all } n \geq n_0 \text{ and } m \geq m_0 \}.
\]

Give corresponding definitions for \( \Omega(g(n, m)) \) and \( \Theta(g(n, m)) \)

Solution:
\[
\Omega(g(n, m)) = \{ f(n, m) : \text{there exist positive constants } c, n_0, \text{ and } m_0 \text{ such that } 0 \leq cg(n, m) \leq f(n, m) \text{ for all } n \geq n_0 \text{ and } m \geq m_0 \}.
\]

\[
\Theta(g(n, m)) = \{ f(n, m) : \text{there exist positive constants } c_1, c_2, n_0, \text{ and } m_0 \text{ such that } c_1g(n, m) \leq f(n, m) \leq c_2g(n, m) \text{ for all } n \geq n_0 \text{ and } m \geq m_0 \}.
\]

Exercise 3.2–1

Show that if \( f(n) \) and \( g(n) \) are monotonically increasing functions, then so are \( f(n) + g(n) \) and \( f(g(n)) \), and if \( f(n) \) and \( g(n) \) are in addition nonnegative, then \( f(n) \cdot g(n) \) is monotonically increasing.

Solution:

(a)
We must show that if \( f(n) \) and \( g(n) \) are monotonically increasing functions, then so is \( f(n) + g(n) \). Suppose not. Let \( n_1 < n_2 \) and \( f(n_1) + g(n_1) > f(n_2) + g(n_2) \). Now, \( f(n_1) \leq f(n_2) \) and \( g(n_1) \leq g(n_2) \)

\[
\begin{align*}
f(n_1) & \leq f(n_2) \\
f(n_1) + g(n_1) & \leq f(n_2) + g(n_2)
\end{align*}
\]

This contradicts our assumption.

(b)
We must show that if \( f(n) \) and \( g(n) \) are monotonically increasing functions, then so is \( f(g(n)) \). Suppose not. Let \( n_1 < n_2 \) and \( f(g(n_1)) > f(g(n_2)) \). Let \( m_1 = g(n_1), m_2 = g(n_2) \). \( m_1 \leq m_2 \). Clearly,
\[ f(m_1) \leq f(m_2), \text{ which contradicts our assumption.} \]

(c)

We must show that if \( f(n) \) and \( g(n) \) are monotonically increasing functions, then so is \( f(n)g(n) \) if \( f \) and \( g \) are nonnegative. Suppose not. Let \( n_1 < n_2 \) and \( f(n_1) + g(n_1) > f(n_2) + g(n_2) \). Now, \( f(n_1) \leq f(n_2) \) and \( g(n_1) \leq g(n_2) \).

\[
\begin{align*}
  f(n_1) & \leq f(n_2) \\
  f(n_1)g(n_1) & \leq f(n_2)g(n_1) \\
  f(n_1)g(n_1) & \leq f(n_2)g(n_2)
\end{align*}
\]

This contradicts our assumption. Note that 4 and 5 hold since \( f \) and \( g \) nonnegative,