VIII. Relations

A. Binary and n-ary relations

1. We have already discussed the concept of a function, which is a special kind of relationship between two sets. Here, we generalize the concept and discuss a variety of possible uses.

2. Recall that $A \times B$ stands for the ordered pairs $(a, b)$, where the first component $a$ is an element of set $A$ and the second component $b$ is an element of set $B$.

Definition: Let $A$ and $B$ be sets, a binary relation from $A$ to $B$ is a subset of $A \times B$.

We write:

\[ a R b \iff (a, b) \in R \]
\[ a R b \iff (a, b) \notin R \]

a. Example: Let $A$ be the set of all US cities and let $B$ be the set of all US states. Hence, $A \times B$ includes: (Atlanta, Arkansas), (Atlanta, New York), ..., (Atlanta, Georgia), ..., (Atlanta, Washington), (Boston, Arkansas), ..., (Boston, Washington), ..., and many, many more.

We specify $R$ by demanding that $(a, b) \in R$ only if city $a$ is in state $b$. 

Note that a relation is defined to be a set.
Thus, we find that (Atlanta, Georgia) is in $R$ and (Atlanta, Arkansas) is not.

6. **Example:** Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Writing $R = \{(0,a), (0,b), (1,a), (2,b)\}$ specifies a relation explicitly by exhaustively writing its elements. We have two ways to represent this relation.

An arrow diagram:

\[ \begin{array}{ccc}
0 & \rightarrow & a \\
1 & \rightarrow & b \\
2 & \rightarrow & a
\end{array} \]

A table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>1</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>

3. **Relations from sets to themselves are of particular importance.**

**Definition:** A relation on the set $A$ is a relation from $A$ to $A$ or a subset of $A \times A$.

a. **Example:** $A = \{1, 2, 3, 4\}$. We define $R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$.

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$. There are relations on a finite set.

b. **We can also define relations on an infinite set.**

**Example:**
(8.3)

\[ R_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \leq b\} \]
\[ R_2 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a = b \lor a = -b\} \]

- \(R_1\) contains \((1, 1), (1, 2), (2, 2)\).
- It does not contain \((2,1)\) or \((1,-1)\).
- Because these pairs violate the constraint \(a \leq b\), \(R_1\) does not contain \((0.5, 1)\).
- Because \((0.5, 1)\) is not in the domain.

Similarly, \(R_2\) contains \((1, 1)\) and \((1, -1)\).
- It does not contain \((0.5, 0.5)\).

4. How many relations is it possible to define on a finite set?

If \(A\) has \(n\) elements, then \(A \times A\) has \(n^2\) elements. Hence, \(A \times A\) has \(2^{n^2}\) subsets, each of which is a relation. Suppose \(n = 2\), e.g., \(A = \{x, y\}\), then there are 16 relations possible:

- \(R_1 = \emptyset\)
- \(R_2 = \{(x, x)\}\)
- \(R_3 = \{(x, y)\}\)
- \(R_4 = \{(y, x)\}\)
- \(R_5 = \{(y, y)\}\)
- \(R_6 = \{(x, x), (x, y)\}\)
- \(R_7 = \{(x, x), (y, x)\}\)
- \(R_8 = \{(x, x), (y, y)\}\)
- \(R_9 = \{(x, y), (y, x)\}\)
- \(R_{10} = \{(x, y), (y, y)\}\)
- \(R_{11} = \{(y, x), (y, y)\}\)
- \(R_{12} = \{(x, x), (x, y), (y, x)\}\)
- \(R_{13} = \{(x, x), (x, y), (y, y)\}\)
- \(R_{14} = \{(x, x), (y, x), (y, y)\}\)
- \(R_{15} = \{(x, y), (y, x), (y, y)\}\)
- \(R_{16} = \{(x, x), (x, y), (y, x), (y, y)\}\)
The number of possibilities grows rapidly. When \( n = 3 \), there are \( 2^{3\cdot3} = 512 \) possible relations.

5. Properties of relations

a. Definition: A relation is **reflexive** if \((a, a) \in R\) for every element \(a \in A\).

Example: \( A = \{x, y\} \quad (n=2)\).
Relations that contain \((x, x)\) and \((y, y)\) are reflexive. Hence, \(R_8, R_{13}, R_{14}, R_{16}\) are reflexive.

Example: The division operator on \( \mathbb{Z} \). Every integer divides itself. Hence, it is reflexive.

b. A relation on a set \( A \) is symmetric if \((b, a) \in R\) when \((a, b) \in R\)

It is antisymmetric if \((a, b) \in R\) and \((b, a) \in R\) implies \(a = b\).

**Note**: These are not opposites!

Example: \( A = \{x, y\} \quad (n=2)\).

\(R_1, R_2, R_5, R_8, R_9, R_{12}, R_{13}, R_{16}\)

are symmetric

\(R_1, R_3, R_7, R_5, R_6, R_7, R_8, R_{10}, R_{11}, R_{13}, R_{14}, R_{18}\)

Missing are: \(R_9, R_{12}, R_{15}, R_{16}\)

Example: The division operator on \( \mathbb{Z} \)
is not symmetric.
Proof: 1/2 but $2 \times 1$ (by example)
The division operator is antisymmetric

C. A relation $R$ on a set $A$ is **transitive** if $(a, b) \in R \land (b, c) \in R
\Rightarrow (a, c) \in R \quad \forall a, b, c \in A$

**Example:** $A = \{a, b, c (n=2)
R_1, R_2, R_3, R_4, R_5$ are vacuously transitive
$R_6, R_7, R_8, R_9, R_{10}, R_{11}, R_{12}, R_{13}, R_{14}$, $R_{15}$ are missing
**Example:** If $a \leq b$ and $b \leq c$ then $a \leq c$
**Proof:** If $a \leq b$, then $\exists m \in \mathbb{Z}^+$ such that $b = ma$. Similarly, if $b \leq c$, then $\exists n \in \mathbb{Z}^+$ such that $c = nb$, Hence $c = (mn)a$ and $mn \in \mathbb{Z}^+$

6. Since relations are subsets of $A \times B$, we can combine them using set operations.

**Example:** Let $A$ be the set of all students and $B$ the set of all courses at UMBC, Hence $A \times B$ contains all students paired with all courses.
Suppose $R_1$ consists of all pairs $(a, b)$ of a student $a$ that has taken course $b$. Suppose $R_2$
Consists all pairs \((a, b)\) of a student \(a\) that requires course \(b\) to graduate.

\(R_1 \cup R_2\) consists of pairs \((a, b)\) where a student \(a\) either has taken a course \(b\) or needs it to graduate.

\(R_1 \cap R_2\) consists of pairs \((a, b)\) where a student \(a\) has taken a course \(b\) that is needed to graduate.

\(R_1 - R_2\) consists of pairs \((a, b)\) where a student \(a\) has taken course \(b\) but does not need it to graduate.

6. Let \(R_1 = \{(x, y) \mid x \in R, x < y\}\)
\(R_2 = \{(x, y) \mid x \in R, x > y\}\)

Then: \(R_1 \cup R_2 = \{(x, y) \mid x \in R, x \neq y\}\)
\(R_1 \cap R_2 = \emptyset\)

7. Composites

**Definition:** Let \(R\) be a relation from a set \(A\) to a set \(B\) and \(S\) a relation from \(B\) to \(C\). The **composite** of \(R\) and \(S\) is the relation consisting of ordered pairs \((a, c)\) where \(a \in A\), \(c \in C\) and for which \(\exists b\), such that \((a, b) \in R\) and \((b, c) \in S\). We denote the composite as \(S \circ R\).
Example: \( R \) relates \( \{1, 2, 3\} \) to \( \{1, 2, 3, 4, 5\} \) with \( R = \{(1,1), (1,4), (2,3), (3,3), (5,5)\}\) 
\( S \) relates \( \{1, 2, 3, 4, 5\} \) to \( \{0, 1, 2, 3\} \) with \( S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}\).

To calculate \( S \circ R \), we first line up the final elements of \( R \) with the initial elements of \( S \):

\[
\begin{array}{ccc}
R & S & S \circ R \\
(1,1) & (1,0) & (1,0) \\
(1,4) & (4,1) & (4,1) \\
(2,3) & (3,2) & (2,2) \\
(3,4) & (4,1) & (3,1) \\
\end{array}
\]

We then obtain the composite elements by putting the pairs together, dropping the middle component. Finally, we eliminate doubles (none here). We obtain: \( S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\} \).

b. One can compose a relation with itself. Example: Let \( R \) be a relation on the set of all people in which \( a \) is a parent of \( b \). Then \( R \circ R \) is the set of pairs \( (a, c) \) where \( a \) is a grandparent of \( c \).

c. We can now recursively define powers of \( R \).
Definition: \( R^n, n \in \mathbb{Z}^+ \) is obtained via \( R^1 = R, \ R^{n+1} = R^n \circ R \)

Example: \( R = \{ (1,1), (2,1), (3,2), (4,3) \} \)
on \( \mathbb{Z} \). Then: \( R^2 = \{ (1,1), (2,1), (3,1), (4,2) \} \)
\( R^3 = \{ (1,1), (2,1), (3,1), (4,3) \} \), \( R^4 = R^3 \)
and so on for \( n \geq 4 \) (\( R^n = R^3 \) for \( n \geq 4 \)).

**a. Theorem:** A relation \( R \) on a set \( A \) is transitive iff \( \forall n \in \mathbb{Z}^+, \ R^n \subseteq R \)

**Proof:** (i) We first prove \( R^n \subseteq R \)
\( \Rightarrow R \) is transitive

If \( R^n \subseteq R \), then \( R^2 \subseteq R \)
If \((a,b) \in R \) and \((b,c) \in R \),
then \((a,c) \in R^2 \) (definition of composition)

Hence \((a,c) \in R \) (hypothesis)
and \( R \) is transitive

(ii) We next prove \( R \) is transitive
\( \Rightarrow R^n \subseteq R \) by induction

(a) \( n = 1 \) : \( R \subseteq R \) by definition

(b) Suppose \( R^n \subseteq R \), then
\( R^{n+1} = R^n \circ R \). Consider all elements \( x \in A \) that
satisfy \((a,x) \in R \) and
\((x,b) \in R^n \). Since \( R^n \subseteq R \),
we have \((x,b) \in R \). By
transitivity \((a,x) \in R \),
\( R \).
8. Definition: Let \( A_1, A_2, \ldots, A_n \) be sets. An \( n \)-ary relation on these sets is a subset of \( A_1 \times A_2 \times \cdots \times A_n \). The sets \( A_1, A_2, \ldots, A_n \) are called the domains of the relation and \( n \) is called the degree.

Example: Let \( R \) be the relation on \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) where \((a, b, c)\) are integers with \( a < b < c \). Then, \((1, 2, 3) \in R\), \((2, 4, 3) \notin R\), \((-1, 2, 3) \notin R\). The domains are all integers with \( n \geq 0 \). The degree is 3.

9. Databases

a. An important application of relations is the manipulation of databases.

A database consists of records made up of fields [records are \( n \)-tuples; fields are the entries of the \( n \)-tuples]

Example: A student table with \( n \)-tuples in the form: student, name, ID-number, major, GPA

(We are following Rosen here)
Student Table:

<table>
<thead>
<tr>
<th>STUDENT-NAME</th>
<th>ID- NUMBER</th>
<th>MAJOR</th>
<th>GPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ackerman</td>
<td>231455</td>
<td>Computer Science</td>
<td>3.88</td>
</tr>
<tr>
<td>Adams</td>
<td>888323</td>
<td>Physics</td>
<td>3.95</td>
</tr>
<tr>
<td>Chou</td>
<td>102147</td>
<td>Computer Science</td>
<td>3.49</td>
</tr>
<tr>
<td>Goodfriend</td>
<td>453876</td>
<td>Mathematics</td>
<td>3.45</td>
</tr>
<tr>
<td>Rao</td>
<td>678543</td>
<td>Mathematics</td>
<td>3.90</td>
</tr>
<tr>
<td>Stevens</td>
<td>786576</td>
<td>Psychology</td>
<td>2.99</td>
</tr>
</tbody>
</table>

6. Definition: A domain of an n-ary relation is called a primary key when it uniquely identifies one element of the relation. 
   [ID- NUMBER is a primary key; MAJOR is not]

Definition: A composite key is a set of domains that uniquely identifies one element of the relation.

10. Queries on databases often consist of asking which elements in the database satisfy constraints that we impose.

   a. Definition: Let R be an n-ary relation and let C be a condition that elements in R may satisfy. Then the selection operator σC maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy C.
Example: \( c_1 \) is MAJOR = computer science

The result is:

<table>
<thead>
<tr>
<th>STUDENT_NAME</th>
<th>ID-NUMBER</th>
<th>MAJOR</th>
<th>GPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ackerman</td>
<td>231455</td>
<td>Computer Science</td>
<td>3.88</td>
</tr>
<tr>
<td>Chou</td>
<td>102147</td>
<td>Computer Science</td>
<td>3.49</td>
</tr>
</tbody>
</table>

Definition: \( b \). The projection \( P_{i_1, i_2, \ldots, i_m} \) maps the \( n \)-tuple \((a_1, a_2, \ldots, a_n)\) to the \( m \)-tuple (where \( m \leq n \)) \((a_{i_1}, a_{i_2}, \ldots, a_{i_m})\).

Example: For our student table, \( P_{i_1, i_4} \) is shown to the left.

<table>
<thead>
<tr>
<th>STUDENT_NAME</th>
<th>GPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ackerman</td>
<td>3.88</td>
</tr>
<tr>
<td>Adams</td>
<td>3.95</td>
</tr>
<tr>
<td>Chou</td>
<td>3.49</td>
</tr>
<tr>
<td>Goodfriend</td>
<td>3.95</td>
</tr>
<tr>
<td>Rao</td>
<td>3.90</td>
</tr>
<tr>
<td>Stevens</td>
<td>2.99</td>
</tr>
</tbody>
</table>

Example: Let \( R \) be a relation of degree \( m \) and let \( S \) be a relation of degree \( n \). The join \( J_p(R \times S) \) where \( p \leq m \) and \( p \leq n \) is a relation of degree \( m + n - p \) that consists of all \((m+p)-\text{tuples}\) \((a_1, a_2, \ldots, a_{m+p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})\) where the \( m \)-tuple \((a_1, a_2, \ldots, a_{m+p}, c_1, c_2, \ldots, c_p)\) belongs to \( R \) and the \( n \)-tuple \((c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})\) belongs to \( S \).

Example:

<table>
<thead>
<tr>
<th>STUDENT_NAME</th>
<th>ID-NUMBER</th>
<th>MAJOR</th>
<th>ADVISOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ackerman</td>
<td>231455</td>
<td>Computer Science</td>
<td>Sherman</td>
</tr>
<tr>
<td>Chou</td>
<td>102147</td>
<td>Computer Science</td>
<td>Pinkston</td>
</tr>
</tbody>
</table>
B. Representing Relations

1. It is important to be able to represent relations in simple ways in order to visualize how the relation works. The first method that we will discuss is the zero-one matrix.

a. Let \( R \) be a relation from \( A = \{a_1, a_2, \ldots, a_m\} \) to \( B = \{b_1, b_2, \ldots, b_n\} \).

We may represent \( R \) using the matrix  
\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
\end{bmatrix}
\]

The matrix is \( 1 \times 1 \).

b. The matrix of a relation \( R \) on \( A \) is square.

1. A relation is **reflexive** iff \((a_i, a_i) \in R \) for \( i = 1, \ldots, n \). Hence, all diagonal elements equal 1.

   **Note:** Opposite elements are not allowed.

2. A relation is **symmetric** iff \((a_i, a_j) \in R \iff (a_j, a_i) \in R \)

   Hence, we must have \( M_{ij} = M_{ji} \).

3. A relation is **anti-symmetric** iff \((a_i, a_j) \in R \land (a_j, a_i) \in R \Rightarrow i = j \)
Examples: Suppose that $R$ is represented by
\[
IM_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]
The relation is reflexive and symmetric, but it is not anti-symmetric.

**Example:**

Set operations

1. $IM_{R_1} \cup IM_{R_2} = IM_{R_1} \lor IM_{R_2} = [m_{i,j} \lor m_{k,j}]$
2. $IM_{R_1} \cap IM_{R_2} = IM_{R_1} \land IM_{R_2} = [m_{i,j} \land m_{k,j}]$

\[
IM_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad IM_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
IM_{R_1 \cap R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

(iii) $IM_{S \circ R} = IM_R \circ IM_S$

where $\circ$ stands for standard Boolean multiplication.

Writing $IM_{S \circ R} = [s_{i,j}]$,

\[
IM_R = [r_{i,j}] \quad \text{and} \quad IM_S = [s_{i,j}],
\]

the pair $(a_i, c_j)$ belongs to $S \circ R$

if for $k$ such that $(a_i, b_{i,k}) \in R$ and $(b_{i,k}, c_j) \in S$. Hence $r_{i,j} = 1$

if $s_{i,k} = s_{i,j} = 1$ for some $k$.

\[
IM_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad IM_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad IM_{S \circ R} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]
Example: $R^T$

$[\begin{array}{c c c c}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{array}]$; $[\begin{array}{c c c c}
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}]$

2. Another approach to representing relations is to use directed graphs or digraphs.

Definition: A directed graph or digraph consists of a set $V$ of vertices or nodes together with a set of ordered elements of $V$ called edges or arcs. The vertex $a$ is called the initial vertex of the edge $(a, b)$ and the vertex $b$ is called the terminal vertex of this edge. An edge of the form $(a, a)$ is called a loop.

a. Examples:

(i) Directed graph of

$R = \{(1,1), (1,3), (2,1), (3,3), (2,4), (3,1), (3,2), (4,1)\}$

(ii) Given the directed $a(i)$, what is $R'$?

$R' = \{(1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,1), (4,3)\}$

b. A relation is reflexive if the digraph has a loop at every vertex.
A relation is **symmetric** iff for every edge between two distinct vertices there is an edge in the opposite direction.

A relation is **anti-symmetric** iff there is at most one edge between two distinct vertices.

A relation is **transitive** iff for every edge from \((x, y)\) and from \((y, z)\), there is an edge from \((x, z)\).

Examples:

(i) reflexive, not symmetric, anti-symmetric or transitive

(ii) not reflexive, symmetric, not anti-symmetric, not transitive
C. Closures

1. Suppose a small airline has flights between a small number of cities in California: \( \text{Los Angeles} \leftrightarrow \text{San Francisco}, \ \text{San Francisco} \leftrightarrow \text{Fresno}, \ \text{Los Angeles} \leftrightarrow \text{San Diego}. \) Can we get from Los Angeles to Fresno? Yes. We fly from Los Angeles to San Francisco and from San Francisco to Fresno. The original relation is non-stop trips. The relation that interests us is all trips. This second relation is the transitive closure of the first.

2. More generally, a relation \( R \) on a set \( A \) may not have a property \( P \). If there is a smallest \( S \) containing \( R \) with property \( P \), then \( S \) is the closure of \( R \). By smallest, we mean that if any set \( T \) has property \( P \) and if \( R \subseteq T \), then \( S \subseteq T \).

3. Reflexive closure: Given a relation \( R \) on a set \( A \), we may define the diagonal relation on \( A \): \( \Delta = \{(a, a) | a \in A\} \). The reflexive closure of \( A \) is given by \( R \cup \Delta = S \).

Proof: By construction \( S \) contains all pairs \( (a, a) \). Hence it must be reflexive. By construction \( R \subseteq S \). Suppose there exists a set \( T \subseteq S \) that is also reflexive.
Either, it does not \( \alpha_R \in R \) or \( \alpha \Delta \in \Delta \).

In the first case, \( T \not\subseteq R \). In the second, \( T \) is not reflexive.

a. Example: \( A = \{1, 2, 3\}, R = \{(1, 1), (1, 2), (2, 1), (3, 2)\} \). \( R \) is not reflexive. In this case, \( \Delta = \{(1, 1), (2, 2), (3, 3)\} \) and \( S = R \cup \Delta = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 3)\} \) is the reflexive closure.

b. Example: \( A = \mathbb{Z}, R = \{(a, b) \mid a < b\} \). In this case, \( \Delta = \{(a, a) \mid a \in \mathbb{Z}\} \) and \( R \cup \Delta = \{(a, b) \mid a \leq b\} \).

4. Symmetric closure: Given a relationship \( R \) on a \( A \), we can define a new relationship \( R' \) on \( A \)
\[
R' = \{(b, a) \mid (a, b) \in R\}
\]
Then, the symmetric closure of \( R \) is given by
\[
S = R \cup R^{-1}
\]
a. Example: \( A = \{1, 2, 3\} \) and \( R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\} \). In this case, \( R^{-1} = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\} \) and \( R \cup R^{-1} = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\} \). We have added two new elements as shown in the Venn diagram to the left.

b. Example: \( A = \mathbb{Z}, R = \{(a, b) \mid a > b\} \). In this case, \( R^{-1} = \{(a, b) \mid a > b\} \) and \( R \cup R^{-1} = \{(a, b) \mid a \neq b\} \).
5. Constructing transitive closures is more difficult! The obvious approach of adding \((a, c)\) whenever \((a, b)\) and \((b, c)\) are in the relation fails.

Example: \(A = \{1, 2, 3, 4\}, \ R = \{(1, 2), (1, 4), (2, 1), (3, 2)\}\). Because \(R\) has \((1, 3)\) and \((3, 2)\), we add \((1, 2)\); \((2, 1) + (1, 2) \rightarrow (2, 2)\); \((3, 2) + (2, 1) \rightarrow (3, 1)\); \((2, 1) + (1, 4) \rightarrow (2, 4)\). We now obtain a new relation:

\[R^1 = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2)\}\]

However, the new relation is still not transitive.

For instance: \((1, 2) + (2, 1) \rightarrow (1, 1)\) must be added.

b. We can continue to iterate:

\[(3, 1) + (1, 2) \rightarrow (2, 2)\]; \((3, 1) + (1, 2) = (3, 3)\); \((3, 1) + (1, 2) = (3, 3)\); \((3, 1) + (1, 2) = (3, 3)\); \((3, 1) + (1, 2) = (3, 3)\); \((3, 1) + (1, 2) = (3, 3)\); \((3, 1) + (1, 2) = (3, 3)\);

which yields

\[R^2 = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}\]

Further iteration yields nothing new.

c. Note: On a finite set \(A\) with \(n\) elements, suppose that a relation \(R\) has \(m\) elements \((m \leq n^2)\). It follows that repeated iteration must produce the transitive closure in at most \(n^2 - m\) steps. Why? \(A \times A\) is transitive, and each iteration must produce one or more new elements of \(A \times A - R\). Since there are only \(n^2 - m\) elements in \(A \times R - R\), the pigeonhole principle implies that we must use them up in \(n^2 - m\) steps or terminate.
d. More efficient procedures and tighter bounds exist. We will discuss them.

6. Directed graphs are helpful in constructing transitive closures. We now introduce some necessary terminology.

a. **Definition**: A **path** from $a$ to $b$ in a digraph $G$ is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ where $n \in \mathbb{N}$, $x_0 = a$, $x_n = b$. The path is denoted $x_0, x_1, \ldots, x_n$ and is said to have length $n$. A path of length $n \geq 1$ that begins and ends at the same vertex is a circuit or a cycle.

**Note**: Paths can pass through the same vertex multiple times.

**Example**: $A = \{a, b, c, d, e\}$

$R = \{(a, a), (a, b), (a, c), (a, e), (b, a), (b, e), (c, b), (d, a), (d, b), (d, c), (d, d), (e, b), (e, c), (e, d)\}$

The digraph has 5 vertices and 14 edges. Are the following paths?

1. $a, b, e, d$
2. $a, e, c, d, b$
3. $b, a, c, b, a, b$
4. $d, c$
5. $c, b, a$
6. $e, b, a, b, a, b, e$

This Fig. on p. 435 of Rosen says "A directed graph".

It is really a representation of a directed graph.
(1) Yes, (2) No; (5, 8) is not on edge
(3) Yes, (4) Yes, (5) Yes, (6) Yes -
Which are circuits? (3) and (6)

6. Theorem: Let \( R \) be a relation on
a set \( A \). There is a path of length
\( n \), where \( n \) is a positive integer,
from \( a \) to \( b \) iff \( (a, b) \in R^n \).
Proof: We will use induction.

Basis step - When \( n = 1 \), there is
a path of length 1 iff \( (a, b) \in R \),
which is true by definition.

Inductive step - Assume that the
theorem is true for \( n \). There
is a path of length \( n+1 \) from
\( a \) to \( b \) iff \( \exists c \) such that \( (a, c) \)
is a path of length 1 and \( (c, b) \)
is a path of length \( n \). By our
basis step and inductive hypothesis,
\( (a, c) \) is a path of length 1 iff \( (a, c) \in R \)
and \( (c, b) \) is a path of length \( n \) iff
\( (c, b) \in R^n \). Hence, \( (a, c) \)
is a path of length \( n+1 \) iff \( (a, c) \in R^{n+1} \).

7. Constructing transitive closures

a. The basic idea is that determining
the transitive closure of a relation is
equivalent to determining which
pairs of vertices in a digraph are
connected by a path.
6. Definition: Let $R$ be a relation on a set $A$. The connectivity relation $R^*$ consists of the pairs $(a, b)$ such that there is a path of at least length 1 from $a$ to $b$.

Since $R^n$ consists of pairs $(a, b)$ such that there is a path of length $n$ from $a$ to $b$, it follows that

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Example: Let $R$ be the relation on subway stops in New York where $(a, b)$ contains stops where it is possible to go from $a$ to $b$ without changing trains. What is $R^n (n \geq 1)$? What is $R^*$?

Solution: $R^n$ is the pairs of stops where it is possible to go from $(a, b)$ with $n-1$ changes. NOTE: It may also be possible to go from $a$ to $b$ in $m$ changes, in which case $(a, b) \in R^n$ and $(a, b) \in R^{m+1}$.

$R^*$ contains the pairs of stops where it is possible to go from $a$ to $b$ in any number of stops.
c. Theorem: The transitive closure of a relation \( R \) equals the connectivity relation \( R^* \).

Proof:
(i) \( R \subseteq R^* \) by definition
(ii) \( R^* \) is transitive: If \((a,b) \in R^*\) and \((b,c) \in R^*\), then there is a path from \( a \) to \( b \) and a path from \( b \) to \( c \) in \( R \). Concatenating these paths, we find that there is a path from \( a \) to \( c \) in \( R \) and hence \((a,c) \in R^*\)
(iii) \( R^* \) is smaller than any other transitive relation containing \( R \).
Suppose \( S \) is a transitive relation containing \( R \). Since \( S \) is transitive, then \( S^n \) is transitive, and \( S^n \subseteq S \) (something we proved earlier). It follows that \( S^* = \bigcup_{k=1}^{\infty} S^k \subseteq S \). We have \( R^* \subseteq S^* \) since any path in \( R \) must be a path in \( S \).
Hence, \( R^* \subseteq S^* \subseteq S \).

d. Lemma: Let \( A \) be a set with \( n \) elements and let \( R \) be a relation on \( A \). If there is a path of length at least one from \( a \) to \( b \), then there is a path of length \( \leq n \).
When \( a \neq b \), this length cannot exceed \( n-1 \).
Proof: Consider a path from \( a \) to \( b \) of length \( m \), where \( a \neq b \) and \( m \geq n \). (The case \( a = b \) should be treated separately.)
Suppose that this path is the shortest path. We label its elements \( x_0, x_1, \ldots, x_m \), where \( x_0 = a \) and \( x_m = b \). Since \( A \) contains \( n \) elements and there are at least \( n+1 \) vertices in the path, at least one must appear twice. In other words, there is some \( i \) and some \( j \), such that \( 0 \leq i < j \leq m \) such that \( x_i = x_j \).
Thus, we may create a new, shorter path by removing the edges \( (x_{i+1}, x_i), (x_{i+1}, x_i), \ldots, (x_{j+1}, x_j) \) which form a cycle. Hence, our path cannot be the shortest. \( \square \)

E. Corollary: \( R^* = R \cup R^2 \cup \cdots \cup R^n \),
In terms of zero-one matrices, we may write:

\[
M_{R^*} = I M_R V I M_R^{[2]} V \cdots V M_R^{[n]} \]
Example: Suppose \( A = \{a, b, c\} \) and \( R = \{(a, a), (a, c), (b, b), (c, a), (c, c)\} \).

The one-zero matrix is:

\[
M_R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix}
\]

We then have \( M_{R^+} = M_R V^T M_{R^2} V M_R \)

\[
= \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix} V \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} V \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} V \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\]

We can encapsulate this observation in the following algorithm (in pseudo-code).

```plaintext
procedure transitive closure (MR: zero-one \( n \times n \) matrix)

A := IMR
B := A
for i := 2 to n
begin
A := A \& MR
B := BMRA
end
{B is the zero-one matrix for \( R^+ \)}
```

What is the complexity of the algorithm?

We have \( n-1 \) Boolean products of \( n \times n \) zero-one matrices. Consider the \( ij \) element of the product, \( A \& B \).

It is given by \((a_{i1} \& b_{1j}) \& (a_{i2} \& b_{2j}) \& \ldots \& (a_{in} \& b_{nj})\). There are \( n \) \( \& \) operations and \( n-1 \) \& operations for a total of \( 2n-1 \) operations.
Thus, for the total product, we have \( n^2 (2n-1) \) operations since there are \( n^2 \) elements. The total number of operations with \( n-1 \) Boolean products is \( n^2 (2n-1) (n-1) = O(n^4) \)

8. Warshall's algorithm (Roy-Warshall algorithm)

a. A more efficient algorithm that is \( O(n^3) \) will now be presented.

1. We use the concept of interior vertices. Definitions: In a path \( x_0, x_1, \ldots, x_m \), the interior vertices are \( x_1, x_2, \ldots, x_{m-1} \)

2. Consider a relation \( R \) on a set \( A \) with \( n \) elements: \( V_1, V_2, \ldots, V_n \)

(1) \( W_0 = M_R \)

(2) \( W_k = [W_j]^k \) where \( [W_j] = 1 \) if there is a path from \( V_i \) to \( V_j \)

such that all the interior vertices of this path are in the set \( \{ V_1, V_2, \ldots, V_k \} \) and is zero otherwise

(3) \( W_n = M_R^n \) since the \( ij \)-th entry of \( W_n \) only equals 1 if there is some path including some number of all the vertices!
We give an example of this construction prior to the general algorithm. Consider $A = \{a, b, c, d\}$ with $R = \{(a, d), (b, a), (b, c), (c, a), (c, d), (d, c)\}$.

We may choose $V_1 = a$, $V_2 = b$, $V_3 = c$, $V_4 = d$.

$$W_0 = IM_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We now proceed as follows to calculate $W_1$:

1. We keep all paths of length 1.
2. We add all paths where $W_{i,1} = 1$ and $W_{i,j} = 1$. In this case, we only have $W_{2,1} = 1$, corresponding to $(b, a)$ and $W_{4,1} = 1$, corresponding to $(c, d)$, and $W_{4,1} = 1$, corresponding to $(a, d)$ and $W_{4,1} = 1$, corresponding to $(c, d)$. Thus, we add the paths $(b, d)$ which is new and $(c, d)$, which is not.

More generally, for each pair $ij$, our new matrix is given by

$$W_{ij}^{(1)} = W_{ij}^{(0)} \lor (W_{i,1}^{(0)} \land W_{1,j}^{(0)})$$

We obtain

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

In the next round, we let

$$W_{ij}^{(2)} = W_{ij}^{(1)} \lor (W_{i,1}^{(1)} \land W_{1,j}^{(1)})$$
In our case $W_{i2}^{(1)} = 0$ for all $i$ (column 2 has all zeros). There are no edges that lead into $b$. It follows that $W_{ij}^{(2)} = W_{ij}^{(1)}$ and $W_2 = W_i$.

In the third round, we let

$$W_{ij}^{(3)} = W_{ij}^{(2)} \lor (W_{ij}^{(2)} \land W_{ij}^{(1)})$$

11: $W_{11}^{(3)} = 0 \lor (0 \land 1) = 0$
12: $W_{12}^{(3)} = 0 \lor (0 \land 0) = 0$
13: $W_{13}^{(3)} = 0 \lor (0 \land 0) = 0$
14: $W_{14}^{(3)} = 1 \lor (0 \land 1) = 1$
21: $W_{21}^{(3)} = 1 \lor (1 \land 1) = 1$
22: $W_{22}^{(3)} = 0 \lor (1 \land 0) = 0$
23: $W_{23}^{(3)} = 1 \lor (1 \land 0) = 1$
24: $W_{24}^{(3)} = 1 \lor (1 \land 1) = 1$
31: $W_{31}^{(3)} = 1 \lor (0 \land 1) = 1$
32: $W_{32}^{(3)} = 0 \lor (0 \land 0) = 0$
33: $W_{33}^{(3)} = 0 \lor (0 \land 0) = 0$
34: $W_{34}^{(3)} = 1 \lor (0 \land 1) = 1$
41: $W_{41}^{(3)} = 0 \lor (1 \land 1) = 1$
42: $W_{42}^{(3)} = 0 \lor (1 \land 0) = 0$
43: $W_{43}^{(3)} = 1 \lor (1 \land 0) = 1$
44: $W_{44}^{(3)} = 0 \lor (1 \land 1) = 1$

In the fourth round, we have

$$W_{ij}^{(4)} = W_{ij}^{(3)} \lor (W_{ij}^{(3)} \land W_{ij}^{(3)})$$
c. In general, when we add \( v_k \) to the list of allowed interior vertices, then there is a path from \( v_i \) to \( v_j \) iff

(i) there is a path from \( v_i \) to \( v_j \) that only involves \( v_i, v_2, \ldots, v_{k-1} \), in which case \( w_{ij}^{(k-1)} = 1 \) or

(ii) there is from \( v_i \) to \( v_k \) that only involves \( v_i, v_2, \ldots, v_{k-1} \) and there is a path from \( v_k \) to \( v_j \) that only involves \( v_i, v_2, \ldots, v_{k-1} \). In this case \( w_{ik}^{(k-1)} \wedge w_{kj}^{(k-1)} = 1 \).

**Lemma:** \( w_{ij}^{(k)} = w_{ij}^{(k-1)} \vee (w_{ik}^{(k-1)} \wedge w_{kj}^{(k-1)}) \)

d. Pseudo-code for Warshall's algorithm

```plaintext
procedure Warshall (MR: n \times n zero-one matrix)
    \( W := MR \)
    for \( k := 1 \) to \( n \)
    begin
        for \( i := 1 \) to \( n \)
        begin
            for \( j := 1 \) to \( n \)
            begin
                \( w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj}) \)
            end
        end
    end
end \{ W = [w_{ij}] is \ MR^k \}
e. Complexity: Each computation $W_{ij}$ requires 2 operations. There are $n^2$ matrix elements and $n$ matrices. So, the total complexity is $2n^3 = \Theta(n^3)$.
D. Equivalence Relations and Partially Ordered Sets

1. It is often useful to use relations to divide a set into non-intersecting subsets and to order the subsets.

   a. Example: Consider a set $A$ that consists of passengers boarding a flight. The passengers in rows 1-10 (first class) board first. After that, the passengers in rows 41-55 board next; next are the passengers in rows 26-40; the passengers in rows 11-25 board last.

We may define the relation $R_1 = \{(a,b) \in A \mid a$ and $b$ board at the same time$\}$.

This relation partitions $A$ into non-overlapping subsets. It is reflexive, transitive, and symmetric. However, it does not tell us anything about the order in which the passengers board.

For that purpose, we define $R_2 = \{(a,b) \in A \mid a$ boards before $b$\}.
This second relation is reflexive, transitive, and anti-symmetric.

2. We may make the concepts that we just introduced concrete through the following definitions.
   a. Definition: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive. Two elements that are related by an equivalence relation are called equivalent.
   b. Definition: A relation R on a set A is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. The set A with its partial ordering R is called a partially ordered set or a poset and is denoted (S, R).

3. Equivalence relations
   a. Example: The relation R on strings of English letters is defined as a R b if $l(a) = l(b)$ [where a and b are strings of English letters]. R is an equivalence relation.
   Proof: (i) reflexivity — a Ra since $l(a) = l(a)$,
   (ii) symmetry — a R b $\iff$ b R a since $l(a) = l(b) \iff l(b) = l(a)$,
   (iii) transitivity — a R b and b R c $\iff$ a R c since $l(b) = l(a)$ and $l(c) = l(a)$ imply $l(c) = l(a)$.
6. Example: Congruence modulo m
Let \( m \in \mathbb{Z}, \geq 2 \). Let \( a \in \mathbb{Z} \) and \( b \in \mathbb{Z} \)
Let \( R = \{(a, b) | a \equiv b \mod m\} \). Then \( R \) is an equivalence relation
Proof: By definition \( a \equiv b \mod m \iff m \mid a-b \).
(i) \( R \) is reflexive. 
(ii) \( m \mid b-a \iff m \mid a-b \).
Hence \( R \) is symmetric. 
(iii) \( m \mid c-b \land m \mid b-a \rightarrow m \mid (c-b)-(b-a) \rightarrow m \mid c-a \).
Hence, \( R \) is transitive. \( \square \)

4. Equivalence classes

a. Definition: Let \( R \) be an equivalence relation on set \( A \). The set of elements in \( A \) that are related to an element is called the equivalence class of \( a \) and is denoted \([a]_R\) or \([a] \) when there is no ambiguity with the relation.
In other words, if \( R \) is an equivalence relation on set \( A \), the equivalence class \([a]_R\) is defined by \([a]_R = \{s \mid (a, s) \in R\} \).
If \( b \in [a]_R \), then \( b \) is called a representative of this class.

b. Example: What are \([0] \) and \([1] \) in the case of congruence modulo 4?
Solution: 
\([0] = \{\ldots, 0 - 2 \cdot 4, 0 - 1 \cdot 4, 0, 0 + 1 \cdot 4, \ldots\} = \{\ldots, -8, -4, 0, 4, 8, 12, \ldots\} \)
\([1] = \{\ldots, 1 - 2 \cdot 4, 1 - 1 \cdot 4, 1, 1 + 4, 1 + 2 \cdot 4, \ldots\} = \{\ldots, -7, -3, 1, 5, 9, 13, \ldots\} \)
More generally, the equivalence classes of the relation congruence modulo m are called congruence classes modulo m. The congruence class of an integer a modulo m is denoted $[a]_m$, so that $[a]_m = \{\ldots, a - 2m, a - m, a, a + m, a + 2m, a + 3m, \ldots\}$.

So, $[0]_7 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$.

5. **Equivalence classes partition a set into disjoint, non-empty subsets.**

To demonstrate this point, we begin with a theorem.

**Theorem:** Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

(i) $a R b$,
(ii) $[a] = [b]$,
(iii) $[a] \cap [b] \neq \emptyset$.

**Proof:** (i) $\Rightarrow$ (ii): Assume $a R b$. Suppose $c \in [a]$, then $a R c$, since $R$ is symmetric. $c R a$ and since $R$ is transitive, $c R b$.

Hence, $[a] \subseteq [b]$. We may similarly show $[b] \subseteq [a]$, so that $[a] = [b]$.

(ii) $\Rightarrow$ (iii): Assume $[a] \neq \emptyset$.

Since $R$ is reflexive, it contains $(a, a)$ and is non-empty. Both $[a]$ and $[b]$ must contain $a$ and hence $[a] \cap [b] \neq \emptyset$.

(iii) $\Rightarrow$ (i): Assume $[a] \cap [b] \neq \emptyset$.

Then, there is a $c$ such that $a R c$ and $b R c$. From symmetry, $c R b$ and from transitivity $a R b$. $\blacksquare$
6. The union of all equivalence classes of $R$ is all of $A$ since each element of $a$ must be in an equivalence class $[a]_R$. We write $\bigcup_{a \in A} [a]_R = A$.

It follows from the equivalence of (ii)
and (iii) that $[a] \neq [b] \Rightarrow [a] \cap [b] = \emptyset$.

Hence, the equivalence classes of $A$
form a partition of $A$.

c. Definition: A partition of a set $S$
is a collection of disjoint, non-empty subsets of $S$ that have $S$ as their union. In other words, the collection of subsets $A_i$, i.e. $\mathcal{I}$ (where $\mathcal{I}$ is an index set) forms a partition of $S$
if $A_i \neq \emptyset$ for $i \in \mathcal{I}$, $A_i \cap A_j \neq \emptyset$, when $i \neq j$, and $\bigcup_{i \in \mathcal{I}} A_i = S$.

Example: $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, $A_3 = \{6\}$ is a partition of $S$.

d. Equivalence classes partition a set. The converse is also true. The partition of a set can be used to form an equivalence relation. Elements of this set are equivalent iff they are in the same subset of the partition.
**Theorem:** Let $R$ be an equivalence relation on a set $S$. Then, the equivalence classes of $R$ form a partition of $S$. Conversely, given a partition $\{A_i | i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i$, $i \in I$, as its equivalence classes.

**Proof:** We already showed that equivalence classes form a partition. We now assume that $\{A_i | i \in I\}$ is a partition on $S$. We let $R$ be the relation on $S$ consisting of all pairs $(x, y)$ where $x$ and $y$ are in the same subset $A_i$. This relation is reflexive since $x$ is in the same subset as itself. It is also symmetric and transitive. Hence, it is an equivalence relation.

**Example:** $A_1 = \{1, 2, 3\}, A_2 = \{4, 5\}, A_3 = \{6\}$ is a partition of $S = \{1, 2, 3, 4, 5, 6\}$. What is the corresponding equivalence relation $R$?

**Solution:** $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 3), (3, 1), (3, 2), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6)\}$. 

\[8.35\]
f. Example: What are the sets in the partition of the integers from congruence modulo 4?

Solution:

\[
\begin{align*}
[0]_4 &= \{ \ldots, -8, -4, 0, 4, 8, \ldots \} \\
[1]_4 &= \{ \ldots, -7, -3, 1, 5, 9, \ldots \} \\
[2]_4 &= \{ \ldots, -6, -2, 2, 6, 10, \ldots \} \\
[3]_4 &= \{ \ldots, -5, -1, 3, 7, 11, \ldots \}
\end{align*}
\]

These are disjoint and their union yields all integers.

6. Partially ordered sets (posets).
Recall that the set \( S \) and the relation \( R \) form a poset \((S, R)\) if \( R \) is reflexive, antisymmetric, and transitive.

a. Example: The \( \geq \) relation is a partial ordering of \( \mathbb{Z} \).

Proof:
(i) reflexivity \( - a \geq a \),
(ii) anti-symmetry \( - a \geq b \text{ and } b \geq a \rightarrow a = b \)
(iii) transitivity \( - a \geq b \text{ and } b \geq c \rightarrow a \geq c \)

b. Example: The \( | \) (divisibility) relation is a partial ordering of \( \mathbb{Z}^+ \).

Proof: (i) reflexivity \( - a \mid a \), (ii) anti-symmetry \( - a \mid b \text{ and } b \mid a \rightarrow a = b \), (iii) transitivity \( - a \mid b \text{ and } b \mid c \rightarrow a \mid c \)

c. Example: The inclusion relation \( \subseteq \) is a partial ordering on the power set of a set \( S \).

Proof: (i) reflexivity \( - A \subseteq A \),
(ii) \( A \subseteq B \text{ and } B \subseteq A \rightarrow A = B \), (iii) \( A \subseteq B \text{ and } B \subseteq C \rightarrow A \subseteq C \)
d. \( a \leq b \) denotes \((a, b) \in R\) in poset notation. We use \( a \leq b \) to denote \( a \leq b \land a \neq b \). We may say "\( a \) is less than or equal to \( b \)" for the relation \( a \leq b \).

**NOTE:** It is not the case in a partially ordered set that in general \( a \leq b \) or \( b \leq a \).

**Example:** In \((\mathbb{Z}, 1)\), 2 is not related to 3, since \(2 \not\leq 3\) and \(3 \not\leq 2\).

e. **Definition:** The elements \( a \) and \( b \) of a poset \((S, \leq)\) are called **comparable** if either \(a \leq b\) or \(b \leq a\). Otherwise, they are called **incomparable**.

**Example:** In the poset \((\mathbb{Z}, 1)\), 3 and 9 are comparable since \(3 \leq 9\), while 5 and 7 are incomparable since \(5 \not\leq 7\) and \(7 \not\leq 5\).

f. **Definition:** If \((S, \leq)\) is a poset and every two elements of \( S \) are comparable, then \( S \) is a **totally ordered** or **linearly ordered** set, while \( \leq \) is called a **total order** or **linear order**. \( S \) is also called a **chain**.

**Example:** \((\mathbb{Z}, \leq)\) is totally ordered since \(a \leq b\) or \(b \leq a\) for any \(a, b \in \mathbb{Z}\).

**Example:** \((\mathbb{Z}^+, 1)\) is not totally ordered since \(5 \not\leq 7\) and \(7 \not\leq 5\).
9. **Definition**: $(S, \leq)$ is a well-ordered set if $\leq$ is a total ordering and every non-empty subset of $S$ has a least element.

**Example**: The set of ordered pairs of positive integers $\mathbb{Z}^* \times \mathbb{Z}^+$ with \((a_1, a_2) \leq (b_1, b_2)\) if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (lexicographic ordering) is a well-ordered set. Proof (sketched): (i) In every non-empty subset, there is a least $a_1$ and a least $a_2$. The combination $(a_1, a_2)$ of these least elements is the least element of the subset, (ii) For every pair $(a_1, a_2)$ and $(b_1, b_2)$, either $a_1 < b_1$ or if $a_1 = b_1$, then $a_2 \leq b_2$.

**Example**: The set $(\mathbb{Z}, \leq)$ is not well-ordered because it has no least element.

7. We can use these concepts to extend the principle of induction.

**Theorem**: The principle of well-ordered induction.

Suppose that $S$ is a well-ordered set. Then $P(x)$ is true for all $x \in S$ if

- **Basis step**: $P(x_0)$ is true for the least element of $S$, and
- **Induction step**: For every $y \in S$ if $P(x)$ is true for all $x \leq y$, then $P(y)$ is true.
Proof: Suppose that $P(x)$ is not true for all $x \in S$. Then, there is an element $y \in S$ such that $P(y)$ is false and the set $A = \{ x \in S | P(x) \text{ is false} \}$ is non-empty and has a least element $a$. It follows that $P(a)$ is true for all $x \in S$ with $x \leq a$. The inductive step then implies that $P(a)$ is true, and we have a contradiction with our original hypothesis.

8. Lexicographic order

a. The words in a dictionary are listed in alphabetical order. This ordering is a special case of lexicographic order or the ordering of strings on a set constructed from a partial ordering on a set. We will show how this construction works on any poset.

b. We begin by constructing a partial ordering on the Cartesian product of two posets, $(A_1, \leq_1)$ and $(A_2, \leq_2)$. The lexicographic ordering $\leq$ on $A_1 \times A_2$ is defined by specifying that $(a_1, a_2) \leq (b_1, b_2)$ if $a_1 \leq_1 b_1$ or $a_1 = b_1$ and $a_2 \leq_2 b_2$.
Example: In the poset \((\mathbb{Z} \times \mathbb{Z}, \preceq)\), we construct \(\preceq\) from the usual \(\leq\) operator on \(\mathbb{Z}\). In this case, we have \((3, 4) \preceq (4, 5), (5, 6) \preceq (6, 1), (4, 2) \preceq (4, 4), (5, 1) \preceq (4, 5),\) and \((4, 3) \preceq (4, 2)\).

c. We can extend this concept to the Cartesian product of \(n\) posets \((A_1, \preceq_1), (A_2, \preceq_2), \ldots, (A_n, \preceq_n)\). We define the partial ordering \(\preceq\) on \(A_1 \times A_2 \times \ldots \times A_n\) by
\[(a_1, a_2, \ldots, a_n) \preceq (b_1, b_2, \ldots, b_n)\]
if \(a_i \preceq_i b_i\), or if there is an integer \(i > 0\) such that \(a_i = b_i, \ldots, a_{i-1} = b_{i-1}\), and \(a_{i+1} \preceq_{i+1} b_{i+1}\).
Example: \((1, 2, 3, 5) \preceq (1, 2, 4, 3)\)
where we take 4-tuples from \(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\) and \(d_i = \leq\) from integers.

d. We now extend this concept to strings of unequal length. Consider the strings \(a_1a_2 \ldots a_m\) and \(b_1b_2 \ldots b_n\), where the string elements come from a partially ordered set \(S\). Let \(t = \min(m, n)\).
Then \(a_1a_2 \ldots a_m \preceq b_1b_2 \ldots b_n\) iff
using the component ordering \((a_1, a_2, \ldots, a_t) \preceq (b_1, b_2, \ldots, b_t)\) or
\((a_1, a_2, \ldots, a_t) = (b_1, b_2, \ldots, b_t)\) and \(\min\).
Example: Consider the usual English letters. We may use the standard order of English letters to construct a lexicographic ordering of the strings.

We have:
meat < meet
meet < meeting