V. Counting

A. Multiplication Rule (Product Rule)

1. In order to keep track of possibilities ... create a
2. **Multiplication rule** (Product Rule)

a. Example: Suppose a computer installation has four
and 10 digits = 36 characters of length 4
36^4 = 1,679,616
(4) Number of PENS without repetition
36 \cdot 35 \cdot 34 \cdot 33 = 1,413,720
because we reduce the number of choices by 1 with each choice of letter.
The probability of picking a PEN with no repetitions is
1,413,720 / 1,679,616 \approx 0.842,
(5) Number of iterations in a nested loop
\begin{align*}
&\text{for } i := 1 \text{ to 4} \\
&\quad \text{for } j := 1 \text{ to 3} \\
&\quad \quad n := n+1 \\
&\quad \text{next } j \\
&\text{next } i
\end{align*}
Thus are 4 \cdot 3 = 12

3. When application is difficult
Ann, Bob, Cyd, and Dan (A, B, C, D)
must take the jobs of
president, treasurer, secretary (p, t, s)
Suppose: Ann cannot be president
Cyd or Dan must be secretary
In a constrained problem like this
Instead, one must construct a full tree.
When in doubt, make a tree!

4. Permutations
   a. A permutation is an ordering of $n$ objects, where $n$ is a positive integer.
      Example: Order $a, b, c$ ($n=3$)
      $a b c, a c b, c a b, b a c, b c a, c b a$
      There are six permutations.

Theorem: For any integer $n$ with $n \geq 1$, the number of $n$-tuples
5.5

(3) If the letters in COMPUTEX are randomly arranged, then what is the probability that CO will appear together?

Answer: \( \frac{n(E)}{n(S)} = \frac{5040}{40,320} = \frac{1}{8} \).

5. Permutations of selected objects

a. Definition: An arrangement
for position 2, \( n-2 \) choices for position 3 and so on up to \( n-\( r-1 \) = n-r+1 \) choices for position \( r \).

C. Important Note! It is easier to write
\[
P(n, r) = \frac{n!}{(n-r)!}, \quad \text{but it is}
\]
\[
\ldots
\]
B. The Addition Rule (Sum rule)

Rosen refers to the addition rule as the sum rule. They are the same thing.

1. Theorem: Suppose a finite set $A$ equals the union of $k$ mutually disjoint sets, then

$$n(A) = n(A_1) + n(A_2) + \cdots + n(A_k)$$

Proof: By induction

Step 1: for $k=1$, $n(A) = n(A_1)$

for $k=2$, $n(A) = n(A_1) + n(A_2)$

because the sets are disjoint, so

$$A_1 = \{x_1, x_2, \ldots, x_{n(A_1)}\}$$

$$A_2 = \{x_{n(A_1)+1}, \ldots, x_{n(A_1)+n(A_2)}\}$$

which can be put in 1-1 correspondence with numbers $1$ to $n(A_1) + n(A_2)$

Step 2: suppose it is true for $k=m$,
then for $B = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_m$, we have

$$n(B) = n(A_1) + n(A_2) + \cdots + n(A_m)$$

Since $B$ and $A_m+1$ are mutually disjoint, $n(A) = n(B) + n(A_{m+1})$

from step 1 and the result is proved. 

NOTE: Step 1 here is harder than step 2!

2. Example: Words with three and fewer letters. How many are possible:

- Words of length 1: 26
- Words of length 2: $26^2 = 676$
- Words of length 3: $26^3 = 17,576$

These sets are disjoint, so,

Total words = $26 + 676 + 17,576 = 18,278$
3. Difference rule

Theorem: If $A$ is a finite set and $B \subseteq A$, then $n(A - B) = n(A) - n(B)$

Proof: $A - B$ and $B$ are mutually disjoint. Hence, by the addition rule

$$n(A) = n(A - B) + n(B)$$

The theorem follows from basic algebra $\Box$

Example: How many four-character PINS have repeated symbols?

We showed earlier that

$$n(\text{all PINS}) = 1,679,616$$
$$n(\text{non-repeating PINS}) = 1,413,720$$

By the difference rule

$$n(\text{repeating PINS}) = 265,896$$

The probability of obtaining a repeating PIN is $n(\text{repeating})/n(\text{total})$ = 0.158

Note: One can obtain this by subtraction from 1. More generally:

Corollary: If $S$ is a finite sample space and $A$ is an event in $S$, then

$$P(A^c) = 1 - P(A)$$

Proof: $A^c$ and $A$ are mutually disjoint
and $A^c + A = S$. By the addition rule $P(A^c + A) = P(A^c) + P(A)$ and $P(S) = 1$ by the definition of probability. The result follows.

4. Inclusion/Exclusion Rule

a. Consider what happens when sets overlap

From the addition rule

$n(A) = n(A - A \cap B) + n(A \cap B)$
$n(B) = n(B - A \cap B) + n(A \cap B)$

$n(A \cup B) = n(A - A \cap B) + n(B - A \cap B) + n(A \cap B) = n(A) + n(B) - n(A \cap B)$

(We take into account that elements in $A \cap B$ are counted twice)

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

6. Example 6. How many integers through 1000 are multiples of 3 or 5? How many are not multiples of either?

Let $A =$ multiples of 3, let $B =$ multiples of 5

(i) \textbf{Every third integer through 999 is divisible by 3. Hence } $n(A) = 333$
(2) Every fifth integer through 1000 is divisible by 5. 
Hence \( n(B) = 200 \)

(3) Any number that is divisible by both 5 and 3 is divisible by 15. Every 15th integer through 990 is divisible by 15. 
Hence \( n(A \cap B) = \frac{990}{15} = 66 \)

So, \( n(A \cup B) = 333 + 200 - 66 = 467 \)

It follows that \( n(A^c \cap B^c) = n((A \cup B)^c) = 0 - n(A \cup B) = 533 \)

\( \frac{1000}{1000} \)

C. Example: A professor does a survey of which students know which computer languages.

Clearly an old example!
(2) How many know all three languages?
\[ n(P \cup V \cup F) = n(P) + n(V) + n(F) \]
\[ - n(P \cap V) - n(P \cap F) - n(V \cap F) + n(P \cap V \cap F) \]
\[ \Rightarrow 47 = 30 + 20 + 18 - 9 - 16 + 8 + n(P \cap V \cap F) \]
\[ \Rightarrow n(P \cap V \cap F) = 6 \]

(3) How many students know Pascal and Fortran but not COBOL?
\[ n(P \cap F - C) = n(P \cap F) - n(P \cap F \cap C) \]
\[ = 9 - 6 = 3 \]

How many students know Pascal, but not Fortran and not COBOL
\[ n(P - (F \cup C)) = n(P) - n(P \cap F) - n(P \cap C) + n(P \cap F \cap C) \]
\[ = 30 - 11 - 16 + 6 = 11 \]
C. Combinations

1. Basic Problem: Given a set \( S \) with \( n \) elements, how many subsets of size \( r \) can be chosen from \( S \)?

Each individual subset is called an \( r \)-combination of \( S \).

Definition: Let \( n \) and \( r \) be non-negative integers with \( r \leq n \). An \( r \)-combination of a set of \( n \) elements is a subset of \( r \) of the \( n \) elements. The symbol \( \binom{n}{r} \), read "\( n \) choose \( r \)" denotes the number of subsets of size \( r \) (\( r \)-combinations) that can be chosen from \( S \).
Note that in our example, the selections are unordered, which means that we do not count \{0,1,3\} and \{0,2,3\} as separate. If we did, then the selections are ordered and there are 12. Ordered selections are permutations: 12 = 4!3 = 4!/(4-3)!

3. Relation between permutations and combinations

**Step 1:** Write 2-combinations of \{0,1,2,3\}

- \{0,1\}
- \{0,2\}
- \{0,3\}
- \{1,2\}
- \{1,3\}
- \{2,3\}

**Step 2:** Order the 2-combinations to obtain 2-permutation

- \{0,1\}
- \{0,2\}
- \{0,3\}
- \{1,2\}
- \{1,3\}
- \{2,3\}

To obtain r-permutations, we may first select r-combinations and then order them.

In this case:

\[ P(n,r) = 12 \]

\[ C(n,r) = \frac{P(n,r)}{r!} = \frac{12}{2!} = 6 \]

More generally:

\[ C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} \]
4. Examples

a. Choose a five member team from a group of 12

\[ \binom{12}{5} = \frac{12!}{5!7!} = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792 \]

There are 792 ways to do it.

b. Turn round in this group of 12
c. Suppose now that we have the opposite problem. We have two people labeled C and D who cannot work together on the five marks...
5. More examples

a. How many eight-bit strings have exactly three 1's.
   
   Answer: We must pick three slots into which to put the 1's. Thus, the answer equals the number of (unordered) ways to pick three numbers out of eight. Hence, the answer is \( \binom{8}{3} = 56 \)

b. How many distinguishable orderings
\[ n_1 \text{ are of type 1 and are indistinguishable from each other} \]
\[ n_2 \text{ are of type 2 and are indistinguishable from each other} \]
\[ n_k \text{ are of type } k \text{ and are indistinguishable from each other} \]

and suppose \( n_1 + n_2 + \ldots + n_k = n \).

Then the number of distinct combinations equals:

\[
\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}
\]

This result, which can be proved by induction, shows that the order in which we take the combinations does not matter.

6. Avoid double-counting!
D. Combinations With Repetition

1. Key question: How do we choose $r$ elements without regard to order from a set of $n$ elements if repetition is allowed?

2. NOTE: If order does matter, the answer is $n^r$ from the product rule. We effectively showed this earlier. Since it went by fast, I will repeat. There are $n$ ways to pick the first element. Since we can repeat, there are $n$ ways to pick the second element, and so on through the $r$-th element. Hence, the
d. Example: Pick three numbers from 1, 2, 3, 45.

Equivalent to looking at all increasing triples:

\[ \{1, 1, 1\}, \{1, 1, 2\}, \{1, 1, 3\}, \{1, 1, 45\}, \{1, 1, 25\}, \{1, 1, 35\}, \{1, 1, 34\}, \{1, 1, 44\}, \{1, 2, 25\}, \{1, 2, 35\}, \{1, 2, 34\}, \{1, 2, 44\}, \{1, 2, 24\}, \{1, 2, 34\}, \{1, 2, 33\}, \{1, 2, 43\}, \{2, 2, 25\}, \{2, 2, 35\}, \{2, 2, 34\}, \{2, 2, 44\}, \{2, 2, 24\}, \{2, 2, 33\}, \{2, 2, 43\}, \{2, 2, 32\}, \{2, 3, 35\}, \{2, 3, 34\}, \{2, 3, 43\}, \{2, 3, 32\}, \{2, 3, 42\}, \{2, 3, 31\}, \{2, 3, 41\}, \{2, 3, 21\}, \{2, 4, 43\}, \{2, 4, 42\}, \{2, 4, 41\}, \{2, 4, 31\}, \{2, 4, 21\}, \{2, 4, 11\}, \{3, 3, 35\}, \{3, 3, 34\}, \{3, 3, 43\}, \{3, 3, 32\}, \{3, 3, 42\}, \{3, 3, 31\}, \{3, 3, 41\}, \{3, 3, 21\}, \{3, 4, 43\}, \{3, 4, 42\}, \{3, 4, 41\}, \{3, 4, 31\}, \{3, 4, 21\}, \{3, 4, 11\}, \{4, 4, 43\}, \{4, 4, 42\}, \{4, 4, 41\}, \{4, 4, 31\}, \{4, 4, 21\}, \{4, 4, 11\}, \{4, 5, 53\}, \{4, 5, 52\}, \{4, 5, 51\}, \{4, 5, 41\}, \{4, 5, 31\}, \{4, 5, 21\}, \{4, 5, 11\} \]

We can write these twenty cases symbolically as shown to the left:

\[ \begin{align*}
1 & 2 & 3 & 4 \\
0 & x & x & x & 1 & 1 & 1 \\
1 & x & x & 1 & 1 & 1 \\
2 & x & x & 1 & 1 & 1 \\
3 & x & x & 1 & 1 & 1 \\
4 & x & x & 1 & 1 & 1 \\
5 & x & x & 1 & 1 & 1 \\
6 & x & x & 1 & 1 & 1 \\
7 & x & x & 1 & 1 & 1 \\
8 & x & x & 1 & 1 & 1 \\
9 & x & x & 1 & 1 & 1 \\
10 & x & x & 1 & 1 & 1 \\
11 & x & x & 1 & 1 & 1 \\
12 & x & x & 1 & 1 & 1 \\
13 & x & x & 1 & 1 & 1 \\
14 & x & x & 1 & 1 & 1 \\
15 & x & x & 1 & 1 & 1 \\
16 & x & x & 1 & 1 & 1 \\
17 & x & x & 1 & 1 & 1 \\
18 & x & x & 1 & 1 & 1 \\
19 & x & x & 1 & 1 & 1 \\
20 & x & x & 1 & 1 & 1 \\
\end{align*} \]

We have 3 x's and 3 1's. Every possible combination of the x's and the 1's is represented once and only once.

Thus, the problem is equivalent to calculating \( C(6,3) \equiv \binom{6}{3} \equiv 20 \).

More generally, if we select \( r \) elements with repetition from \( n \) elements, it is equivalent to combining \( n-1 \) 1's with \( r \) x's, which leads to the theorem...
f. Theorem: The number of $r$-combinations with repetition allowed is given by
\[ C(r+n-1, r) = \binom{r+n-1}{r} \]

We have given a non-rigorous proof. A rigorous proof can be made by induction.

2. Example: A person wants to buy 15 cans of soda from a store...
3. Iterations of a loop
E. Two Classic Theorems

1. Pascal's Triangle and Formula

\[
\begin{array}{cccc}
\hline
n \equiv 0 & 1 \\
\hline
n \equiv 1 & 1 & 1 \\
\hline
n \equiv 2 & 1 & 2 & 1 \\
\hline
n \equiv 3 & 1 & 3 & 3 & 1 \\
\hline
n \equiv 4 & 1 & 4 & 6 & 4 & 1 \\
\hline
\end{array}
\]

In constructing the triangle, \( r \leq n \) and \( T(n, r) \).

\[
- T(n-1, r) + T(n-1, r-1)
\]
Examples:

0: \((a+b)^0 = 1\)

1: \((a+b)^1 = a + b\)

2: \((a+b)^2 = a^2 + 2ab + b^2\)

3: \((a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)

4: \((a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\)