Problem Set #5 Solutions

1. Epp #6.1.11:
   a. The three digit numbers run from 100 to 999. The smallest three digit number that is divisible by 6 is 102, and $102 \div 6 = 17$. The largest three digit number that is divisible by 6 is 996, and $996 \div 6 = 166$. Hence, the number of three digit numbers that are divisible by 6 equals the number of integers between 17 and 166, inclusive. This number is $166 - 17 + 1 = 150$.
   
b. There are a total of 900 three digit numbers. So, the probability of randomly choosing a three digit number that is divisible by 6 equals $\frac{150}{900} = \frac{1}{6} \approx 0.167$.

2. Epp #6.2.7:
   a. The tree diagram for this problem is:

   ![Tree Diagram]

   b. There are 24 outcomes of this experiment.
   
c. The probability that two red balls are chosen equals $\frac{8}{24} = \frac{1}{3}$. 
3. Epp #6.3.20:

a. By the multiplication rule, the number of strings of length \( n \) over \( \{a, b, c, d\} \) with no two consecutive characters the same equals \( 4 \cdot 3 \cdot 3 \cdot \cdots \cdot 3 \), where 3 is repeated \( n - 1 \) times. The reason is that the first character may be chosen in four ways. After that, there are only three ways to choose each subsequent character because each character cannot be the same as the one before it. Thus, we conclude that the answer we want is \( 4 \cdot 3^{n-1} \). By the difference rule, we conclude that the number of strings with at least two consecutive characters the same is given by \( 4^n - 4 \cdot 3^{n-1} \), since the total number of strings is \( 4^n \).

b. The probability is given by:

\[
\frac{4^{10} - 4 \times 3^9}{4^{10}} = 1 - \left(\frac{3}{4}\right)^9 \approx 0.925
\]

4. We first calculate the probability that no two people among the thirty people has the same birthday. The first person’s birthday can be selected in 365 ways, the second person’s birthday can be selected in 364 ways and so on. Hence, the number of ways of selecting the birthdays for the 30 people is given by \( 365!/(365 - 30)! = 365!/335! \). To get the probability that we want, we note that the total number of ways of selecting the birthdays is given by \( 365^{30} \), so that the total probability of selecting 30 people with no birthdays in common is

\[
\frac{365!}{365^{30} \cdot 336!}
\]

Each of the numbers in this result is very large and computing any of them directly is difficult and will lead to a loss of numerical accuracy in the final result. The right way to get the answer is to write it in the form:

\[
\frac{365 \cdot 364 \cdot \cdots \cdot 336}{365 \cdot 365 \cdot \cdots \cdot 365} \approx 1.0000 \cdot 0.9973 \cdot \cdots \cdot 0.9205
\]

That way, we are multiplying together a string of numbers that are all close to 1. The result, which I obtained using MATLAB, is 0.2937. Then, using the subtraction rule, the probability that two or more people will have the same birthday in a group of 30 equals \( 1 - 0.2937 = 0.7063 \) or close to 70%
5. **Epp #6.5.4:**

The answers are given by

a. 
\[
C(30 + 10 - 1, 30) = \binom{30 + 10 - 1}{30} = \binom{39}{30} = 211,915,132
\]

b. 
\[
C(26 + 10 - 1, 26) = \binom{26 + 10 - 1}{26} = \binom{35}{26} = 70,607,460
\]

c. 
\[
\frac{C(35, 26)}{C(39, 30)} \approx 0.3332 = 33.32\%
\]

d. 
\[
\frac{C(26 + 9 - 1, 26)}{C(39, 30)} = \frac{18,156,204}{211,915,132} \approx 0.0857 = 8.57\%
\]

6. **Epp #6.7.17:**

a. Let \( n \) be an integer with \( n \geq 0 \). Apply the binomial theorem with \( a = 1 \) and \( b = x \) to obtain

\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k
\]

because any power of 1 is 1.

b. Taking the derivative of both sides of the equation in part (a), we find

\[
n(1 + x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} kx^{k-1}
\]

c(i). We substitute \( x = 1 \) into the result of part (b) to obtain,

\[
n(1 + 1)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k = \binom{n}{1} \cdot 1 + \binom{n}{2} \cdot 2 + \cdots + \binom{n}{n} \cdot n
\]

Dividing both sides by \( n \) and simplifying yields,

\[
2^{n-1} = \left[ \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} \right]
\]

c(ii). Let \( n \) be an integer with \( n \geq 1 \). Apply the formula from part (b) with \( x = -1 \) to obtain

\[
0 = n[1 + (-1)]^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k(-1)^{k-1}
\]
Problem 6 continued:

d. Apply the formula from part (b) with $x = 3$ to obtain

$$n \cdot 4^{n-1} = n(1 + 3)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k \cdot 3^{k-1} = \frac{1}{3} \sum_{k=1}^{n} \binom{n}{k} k \cdot 3^{k}$$

We conclude that

$$\sum_{k=1}^{n} \binom{n}{k} = 3 \cdot n \cdot 4^{n-1}$$