Problem Set #4 Solutions

1. Epp #7.1.2–7.1.3:

7.1.2. a. domain of \( g = \{1, 3, 5\} \), co-domain of \( g = \{s, t, u, v\} \);

b. \( g(1) = g(3) = g(5) = t \);

c. range of \( g = t \);

d. inverse image of \( t = \{1, 3, 5\} \), inverse image of \( u = \emptyset \);

e. \( \{(1, t), (3, t), (5, t)\} \)

7.1.3. The arrow diagram in \( d \) determines a function; those in \( a, b, c, \) and \( e \) do not.

2. Epp #4.1.34–4.1.38:

4.1.34. \( \prod_{j=1}^{4} (1 - r^j) \)

4.1.35. \( \sum_{k=1}^{n} k^3 \)

4.1.36. \( \sum_{k=1}^{n} \frac{k}{(k+1)!} \)

4.1.37. \( \sum_{i=0}^{n-1} (n-i) \)

4.1.38. \( \sum_{k=0}^{n-1} \frac{n-k}{(k+1)!} \)

3. Epp #4.1.57:

a. Proof: Let \( n \) be an integer such that \( n \geq 2 \). By the definition of the factorial,

\[
 n! = \begin{cases} 
 2 \cdot 1 & \text{if } n = 2 \\
 3 \cdot 2 \cdot 1 & \text{if } n = 3 \\
 n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 & \text{if } n > 3
\end{cases}
\]

In each case, \( n! \) has a factor of 2, and so

\( n! = 2 \cdot k \) for some integer \( k \).

Then,

\[
 n! + 2 = 2 \cdot k + 2 \quad \text{(by substitution)}
\]

\[
 = 2 \cdot (k + 1) \quad \text{(by factoring out the 2)}
\]

Since \( k + 1 \) is an integer, \( n! + 2 \) is divisible by 2. Q.E.D.
Problem 3 continued:

b. Proof: Let \( n \) and \( k \) be integers with \( n \geq 2 \) and \( 2 \leq k \leq n \). Now, \( n! \) is the product of all the integers from 1 to \( n \), and so since \( n \geq 2 \) and \( 2 \leq k \leq n \), \( k \) is a factor of \( n! \). This result implies that \( n! = k \cdot r \) for some integer \( r \). By substitution, \( n! + k = k \cdot r + k = k(r + 1) \). But \( r + 1 \) is an integer since \( r \) is an integer. By the definition of divisibility, it follows that \( n! + k \) is divisible by \( k \). Q.E.D.

c. Yes. If \( m \) is any integer that is greater than or equal to 2, then none of the terms of the following sequence of integers is prime: \( m! + 2 \), \( m! + 3 \), \( m! + 4 \), \ldots, \( m! + m \). The reason is that each has the form \( m! + k \) for an integer \( k \) with \( 2 \leq k \leq m \), and for each such \( k \), by part (b), \( m! + k \) is divisible by \( k \).

4. Epp #4.2.10:

The formula is true for \( n = 0 \): The formula holds for \( n = 1 \) because

\[ 1^3 = \left( \frac{1(1+1)}{2} \right)^2 \]

If the formula is true for \( n = k \), then it is true for \( n = k + 1 \): Suppose

\[ 1^3 + 2^3 + \ldots + k^3 = \left( \frac{k(k+1)}{2} \right)^2 \]

for an integer \( k \geq 1 \). We must show that

\[ 1^3 + 2^3 + \ldots + (k+1)^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2 \]

By the laws of algebra and substitution from the inductive hypothesis

\[ 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \left( \frac{k(k+1)}{2} \right)^2 + (k+1)^3 \]

\[ = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \]

\[ = \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \left( \frac{(k+1)(k+2)}{2} \right)^2 \]

Q.E.D.
5. Epp #4.3.5:

Writing

\[ S_n = \sum_{k=1}^{n} \frac{k}{(k+1)!}, \]

we find

\[ S_1 = \frac{1}{2} = \frac{1}{2}, \quad S_2 = \frac{1}{2} + \frac{2}{3!} = \frac{5}{6}, \quad S_3 = \frac{5}{6} + \frac{3}{4!} = \frac{23}{24}, \]
\[ S_4 = \frac{23}{24} + \frac{4}{5!} = \frac{119}{120}, \quad S_5 = \frac{119}{120} + \frac{5}{6!} = \frac{719}{720}, \]

We see that for \( S_1 \sim S_5 \), we have the relationship,

\[ S_n = \frac{(n+1)! - 1}{(n+1)!}. \]

We thus conjecture that this relationship holds for any \( n \).

*Proof by mathematical induction:*

**The formula holds for** \( n = 1 \): We already showed that it holds for \( S_1 \).

**If the formula holds for** \( n = r, \ r \geq 1 \), **then it holds for** \( n = r + 1 \): From the definition of \( S_n \), it follows that

\[ S_{r+1} = S_r + \frac{r+1}{(r+2)!}, \]

and, from the inductive hypothesis, we may rewrite this equation as

\[ S_{r+1} = \frac{(r+1)! - 1}{(r+1)!} + \frac{(r+1)}{(r+2)!} \]

Using basic algebra, we now find

\[ S_{r+1} = \frac{(r+1)! - 1 (r+2)}{(r+1)!} + \frac{(r+1)}{(r+2)!} \]
\[ = \frac{(r+2)! - (r+2)}{(r+2)!} + \frac{(r+1)}{(r+2)!} \]
\[ = \frac{(r+2)! - 1}{(r+2)!} \]

The last line,

\[ S_{r+1} = \frac{(r+2)! - 1}{(r+2)!}, \]

is what we needed to show. Q.E.D.
6. **Epp #4.3.25:**

*Proof by mathematical induction:*

**The formula holds for** \( n = 1 \): In this case, the formula just asserts

\[
\frac{1}{3} = \frac{1}{3},
\]

which is evidently true.

**If the formula holds for** \( n = k \), **then it holds for** \( n = k + 1 \): Our inductive hypothesis is

\[
\frac{1}{3} = \frac{1 + 3 + \ldots + (2k - 1)}{(2k + 1) + (2k + 3) + \ldots + (4k - 1)},
\]

or, equivalently,

\[
\sum_{i=1}^{k} (2k + 2i - 1) = 3 \sum_{i=1}^{k} (2i - 1) \quad (1)
\]

We must prove

\[
\frac{1}{3} = \frac{1 + 3 + \ldots + (2k + 1)}{(2k + 1) + (2k + 3) + \ldots + (4k + 1)},
\]

or, equivalently,

\[
\sum_{i=1}^{k+1} (2(k + 1) + 2i - 1) = 3 \sum_{i=1}^{k+1} (2i - 1) \quad (2)
\]

Noting that we may write \( 2k = 2 \cdot (\text{the sum of } k \text{ ones}) \), we may write

\[
6k + 3 = 2 \cdot \sum_{i=1}^{k} (1) + 4k + 3
\]

Adding \( 6k + 3 \) to both sides of (1) and using this relationship, we obtain

\[
\sum_{i=1}^{k} (2k + 2i - 1) + 2 \sum_{i=1}^{k} (1) + 4k + 3 = 3 \sum_{i=1}^{k} (2i - 1) + 3(2k + 1)
\]

Combining the sums on the left-hand side of this equation and re-writing the additional terms, we find

\[
\sum_{i=1}^{k} (2(k + 1) + 2i - 1) + 4(k + 1) - 1 = 3 \sum_{i=1}^{k} (2i - 1) + 3(2(k + 1) - 1)
\]

The additional terms in this last equation just equal the \( k + 1 \)-th terms in the sum in (2). Hence, this last equation implies (2), which is what we had to prove. Q.E.D.
7. Epp #4.4.8:

a. Let $P(n)$ be the inequality $h_n \leq 3^n$. We prove by strong mathematical induction that the inequality is true for all integers $n \geq 0$

**The inequality is true for $n = 0, 1, \text{ and } 2$:** Note that $h_0 = 1 \leq 3^0$, $h_1 = 2 \leq 3^1$, and $h_2 = 3 \leq 3^2$.

**If the inequality is true for all integers $i$ with $i \leq k$, then it is true for $k$:**

Let $k$ be an integer with $k > 2$, and suppose the inequality holds for all integers $i$ with $0 \leq i \leq k$. We must show that the inequality is true for $k$. By definition, we have

$$h_k = h_{k-1} + h_{k-2} + h_{k-3},$$

from which we infer that

$$h_k \leq 3^{k-1} + 3^{k-2} + 3^{k-3} = 3^{k-3}(3^2 + 3 + 1) = 3^{k-3} \cdot 13$$

by the inductive hypothesis. It follows that

$$h_k \leq 3^{k-3} \cdot 13 \leq 3^{k-3} \cdot 27 = 3^k,$$

which is what we needed to show. Q.E.D.

b. Let $s$ be any real number such that $s^3 \geq s^2 + s + 1$, and let $P(n)$ be the inequality $h_n \leq s^n$. We prove by strong mathematical induction that this inequality is true for all integers $n \geq 2$.

**The inequality is true for $n = 2, 3, \text{ and } 4$:** Because $s \geq 1.83$, $h_2 = 3 \leq 3.3489 = 1.83^2 < s^2$, $h_3 = h_0 + h_1 + h_2 = 1 + 2 + 3 = 6 \leq 6.1285 = 1.83^3$, and $h_4 = h_1 + h_2 + h_3 = 2 + 3 + 11 \leq 11.2151 = 1.83^4$.

**If the inequality is true for all integers $i$ with $i \leq k$, then it is true for $k$:**

Let $k$ be an integer with $k > 4$, and suppose that the inequality is true for all integers with $0 \leq i \leq k$. We must show that the inequality is true for $k$. By definition, we have

$$h_k = h_{k-1} + h_{k-2} + h_{k-3},$$

from which we infer that

$$h_k \leq s^{k-1} + s^{k-2} + s^{k-3} = s^{k-3}(s^2 + s + 1)$$

by the inductive hypothesis. Since by this same hypothesis, $s^2 + s + 1 \leq s^3$, it follows that

$$h_k \leq s^k,$$

which is what we needed to show. Q.E.D.
8. Epp #4.5.7:

I. Basis Property: $I(0)$ is the statement:

\[
largest = \text{the value of } A[1]\]

and it follows from the pre-condition that this statement is true.

II. Inductive Property: Suppose $k$ is a non-negative integer such that $G \land I(k)$ is true before an iteration of the loop. Then as execution reaches the top of the loop, $i \neq m$, the guard is passed and statement 1 is executed. Now, before the execution of statement 1, $i_{\text{old}} = k + 1$. So, after execution of statement 1, $i_{\text{new}} = i_{\text{old}} + 1 = k + 2$. Also, before statement 2 is executed, $largest_{\text{old}} = \text{the maximum value of } A[1], A[2], \ldots, A[k+1]$. Statement 2 checks whether $A[i_{\text{new}}] = A[k+2] > largest_{\text{old}}$. If the condition is true, then $largest_{\text{new}}$ is set equal to $A[k+2]$, which is the maximum value of $A[1], A[2], \ldots, A[k+1], A[k+2]$. If the condition is false, then $A[k+2] \leq largest_{\text{old}}$ and so $largest_{\text{old}}$ is the maximum value of $A[1], A[2], \ldots, A[k+1], A[k+2]$. In this case, we set $largest_{\text{new}}$ equal to $largest_{\text{old}}$. In either case, $largest_{\text{new}}$ equals the maximum value of $A[1], A[2], \ldots, A[k+1], A[k+2]$. Hence, $I(k+1)$ is true.

III. Eventual Falsity of the Guard: The guard $G$ is the condition $i \neq m$. By I and II, it is known that for all iterations of the loop $I(n)$ is true. Hence, after $m-1$ iterations of the loop, $I(m)$ is true and $G$ is false.

IV. Correctness of the Post-Condition: Suppose that $N$ is the least number of iterations after $G$ is false and $I(n)$ is true. In this case, we have from III that $N = m - 1$ and from I and II that $i = m$ and $largest = \text{the maximum value of } A[1], A[2], \ldots, A[m]$. Hence, the post-condition is satisfied.