Examination #2 Solutions

1. We will refer to the \( n \)-th sum as \( S_n \). Looking at values for \( n \leq 4 \), we find \( S_1 = 1/2 \), \( S_2 = 3/4 \), \( S_3 = 7/8 \), \( S_4 = 15/16 \). From this pattern, we may readily conjecture that \( S_n = (2^n - 1)/2^n \). We now prove it using induction. We know that this conjecture is true when \( n = 1, 2, 3, \) and \( 4 \). Assuming that it is true when \( n = k \), we will show that it is true for \( n = k + 1 \). Writing \( S_n = \sum_{j=1}^{n}(1/2^j) \), we have

\[
S_{k+1} = S_k + \frac{1}{2^{k+1}} = \frac{(2^k - 1)}{2^k} + \frac{1}{2^{k+1}} \quad \text{(where we used the inductive hypothesis)}
\]

\[
= \frac{(2^{k+1} - 2) + 1}{2^{k+1}} = \frac{(2^{k+1} - 1)}{2^{k+1}},
\]

which proves our result.

2. We operate recursively on the size of the list. If there is only one element, then it is the smallest. Otherwise, we find the smallest element in the list consisting of all but the last element of our original list, and compare it to the last element of the original list. Whichever is the smallest is the answer. We assume that there is already a function \( \text{min} \), defined for two arguments that produces the smaller. Here is the procedure in pseudo-code:

```plaintext
procedure smallest(a_1, a_2, \ldots, a_n: integers)
if n = 1 then smallest(a_1, a_2, \ldots, a_n) := a_1
else smallest(a_1, a_2, \ldots, a_n) := min(smallest(a_1, a_2, \ldots, a_{n-1}), a_n)
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3. There are four possible remainders when we divide any integer by four (0, 1, 2, and 3). These are the pigeonholes in this problem. Therefore, by the pigeonhole principle, at least two of the five remainders must be the same.

4. a. Each flip can be either heads or tails; so, there are \( 2^{10} = 1024 \) possible outcomes.

b. To specify an outcome that has exactly two heads, we must choose the two flips that came up heads. There are \( C(10, 2) = 45 \) such outcomes.

c. To contain at most three tails means to contain three tails, two tails, one tail, or no tails. Reasoning as in part (b), we see that that are \( C(10, 3) + C(10, 2) + C(10, 1) + C(10, 0) = 120 + 45 + 10 + 1 = 176 \) such outcomes.

d. To have an equal number of heads and tails in this case means to have five heads. Therefore, the answer is \( C(10, 5) = 252 \).
5. We will solve this problem by first separately calculating the expectations for the points in the true/false category and the points in the multiple choice category. Since the expectation of the sum equals the sum of the expectations, we will add these two expectations once we have computed them. Let $X$ be the expectation for the number of points in the true/false category. Each question is a Bernoulli trial with a success rate of 0.9. Hence, the expectation is $50 \cdot 0.9 \cdot 2 = 90$, or we may write $E(X) = 90$. Similarly, for the multiple choice questions, we find that $E(Y) = 25 \cdot 0.8 \cdot 4 = 80$. We conclude that $E(X + Y) = E(X) + E(Y) = 90 + 80 = 170$. 