An Intuitive Overview of the Theory of Quantum Knots

Samuel Lomonaco
University of Maryland Baltimore County (UMBC)
Email: Lomonaco@UMBC.edu
WebPage: www.csee.umbc.edu/~lomonaco

This is work in collaboration with Louis Kauffman

PowerPoint Lectures and Exercises can be found at:

www.csee.umbc.edu/~lomonaco

This talk is based on the paper:

Lomonaco and Kauffman, Quantum Knots and Lattices, to appear soon on quant-ph

This talk was motivated by:

Lomonaco and Kauffman, Quantum Knots and Mosaics, Journal of Quantum Information Processing, vol. 7, Nos. 2-3, (2008), 85-115. An earlier version can be found at:
http://arxiv.org/abs/0805.0339
This talk was also motivated by:

- Lomonaco, Samuel J., Jr., The modern legacies of Thomson’s atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 – 166.

Throughout this talk:

- “Knot” means either a knot or a link

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**Preamble**

**Objectives**

- We seek to create a quantum system that simulates a closed knotted physical piece of rope.
- We seek to define a quantum knot in such a way as to represent the state of the knotted rope, i.e., the particular spatial configuration of the knot tied in the rope.
- We also seek to model the ways of moving the rope around (without cutting the rope, and without letting it pass through itself.)

**Rules of the Game**

Find a mathematical definition of a quantum knot that is

- Physically meaningful, i.e., physically implementable, and
- Simple enough to be workable and useable.

**Thinking Outside the Box**

**Quantum Mechanics**

is a tool for exploring

**Knot Theory**
Aspirations

We would hope that this definition will be useful in modeling and predicting the behavior of knotted vortices that actually occur in quantum physics such as:

- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

Themes

\[
\begin{align*}
\text{Knot Theory} & \quad = \quad \text{Formal Rewriting System} \\
\text{Formal Rewriting System} & \quad = \quad \text{Group Representation} \\
\text{Quantum Mechanics} & \quad = \quad \text{Group Representation Theory}
\end{align*}
\]

Overview

- Preamble
- Mosaic Knots
- Quantum Mechanics: Whirlwind Tour
- Quantum Knots & Quantum Knot Systems via Mosaics
- Preamble to Lattice Knots
- Lattice Knots
- Q. Knots & Q. Knot Systems via Lattices
- Future Directions & Open Questions

Mosaic Knots

Transforming Knot Theory into a formal Rewriting System

Mosaic Tiles

Let \( T^{(u)} \) denote the following set of 11 symbols, called mosaic (unoriented) tiles:

Please note that, up to rotation, there are exactly 5 tiles

Mosaic Knots

A 4-mosaic trefoil

Figure Eight Knot 5-Mosaic

Let $K(n)$ = the set of $n$-mosaic knots

Hopf Link 4-Mosaic

Sub-Mosaic Moves

A Cut & Paste Move
We will now re-express the standard moves on knot diagrams as sub-mosaic moves.

Planar Isotopy Moves as Sub-Mosaic Moves

It is understood that each of the above moves depicts all moves obtained by rotating the 2x2 sub-mosaics by 0, 90, 180, or 270 degrees.

For example, represents each of the following 4 moves:
Planar Isotopy (PI) Moves on Mosaics

Each of the PI 2-submosaic moves represents any one of the \((n-2+1)^2\) possible moves on an \(n\)-mosaic.

Reidemeister Moves
as
Sub-Mosaic Moves

Reidemeister (R) Moves on Mosaics

Each PI move and each R move is a permutation of the set of all knot \(n\)-mosaics \(K^{(n)}\).

In fact, each PI and R move, as a permutation, is a product of disjoint transpositions.

The Ambient Group \(A(n)\)

We define the ambient group \(A(n)\) as the subgroup of the group of all permutations of the set \(K^{(n)}\) generated by the all PI moves and all Reidemeister moves.
The Mosaic Injection $\iota : M^{(n)} \to M^{(n+1)}$

We define the mosaic injection $\iota : K^{(n)} \to K^{(n+1)}$

$$K^{(n+1)}_{i,j} = \begin{cases} K^{(n)}_{i,j} & \text{if } 0 \leq i, j \leq n \\ \# & \text{otherwise} \end{cases}$$

Mosaic Knot Type

**Def.** Two $n$-mosaic knots $K$ and $K'$ are of the same knot $n$-type, written $K^n \sim K'^n$, provided there exists an element of the ambient group $A(n)$ that transforms $K$ into $K'$.

Two $n$-mosaics $K$ and $K'$ are of the same knot type if there exists a non-negative integer $k$ such that $\iota^k K^n \sim \iota^k K'^n$.
Conjecture: The Mosaic Knots formal rewriting system fully captures tame knot theory.

Recently, Takahito Kuriya has proven this conjecture,


Quantum Mechanics

The state of a Quantum System is a vector $|\psi\rangle$ (pronounced ket $\psi$) in a Hilbert space $\mathcal{H}$.

**Def.** A Hilbert Space is a vector space $\mathcal{H}$ over $\mathbb{C}$ together with an inner product $\langle -,- \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ such that

1) $\langle u+u',v \rangle = \langle u,v \rangle + \langle u',v \rangle$ & $\langle \lambda u,v \rangle = \lambda \langle u,v \rangle$

2) $\langle u,v \rangle = \langle v,u \rangle$

3) Cauchy seq $u_n$ in $\mathcal{H}$, $\lim_{n \to \infty} u_n \in \mathcal{H}$
The dynamic behavior of a quantum system is determined by Schrödinger’s equation.

\[ \frac{i}{\hbar} \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \]

where \( t \) is time, and where \( U(t) \) is a curve in the group \( \mathcal{U}(\mathcal{H}) \) of unitary transformations on the state space \( \mathcal{H} \).

An observable \( \Omega \) is a Hermitian operator on the state space \( \mathcal{H} \), i.e., a linear transformation such that

\[ \Omega^\dagger = \Omega^T = \Omega \]

The Hilbert space \( \mathcal{K}^{(n)} \) of quantum knots is the Hilbert space with the set \( \mathcal{K}^{(n)} \) of \( n \)-mosaic knots as its orthonormal basis, i.e., with orthonormal basis

\[ \{ |K\rangle : K \in \mathcal{K}^{(n)} \} \]

Recall that \( \mathcal{K}^{(n)} \) is the set of all \( n \)-mosaic knots.

For Q.M. systems, we need an underlying Hilbert space. So we define:

The Hilbert space \( \mathcal{K}^{(n)} \) of quantum knots is the Hilbert space with the set \( \mathcal{K}^{(n)} \) of \( n \)-mosaic knots as its orthonormal basis, i.e., with orthonormal basis

\[ |K\rangle = \frac{1}{\sqrt{2}} (|K_1\rangle + |K_2\rangle) \]
Since each element is a permutation, it is a linear transformation that simply permutes basis elements. Hence, under this identification, the ambient group becomes a discrete group of unitary transformations on the Hilbert space .

**The Quantum Knot System** (, )

Def. A quantum knot system (, ) is a quantum system having as its state space, and having the Ambient group as its set of accessible unitary transformations. The states of quantum system (, ) are quantum knots. The elements of the ambient group are quantum moves.

Choosing an integer is analogous to choosing a length of rope. The longer the rope, the more knots that can be tied.

The parameters of the ambient group are the "knobs" one turns to spatially manipulate the quantum knot.

**Quantum Knot Type**

Def. Two quantum knots and are of the same knot -type, written if,

provided there is an element such that

They are of the same knot type, written if,

provided there is an integer such that

Two Quantum Knots of the Same Knot Type
Two Quantum Knots NOT of the Same Knot Type

\[ |K_1\rangle = \begin{array}{c}
\end{array} \]

\[ |K_2\rangle = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \]

\[ \sqrt{2} \]

Hamiltonians of the Generators of the Ambient Group

Hamiltonians for \( A(n) \)

Each generator \( g \in A(n) \) is the product of disjoint transpositions, i.e.,

\[ g = (K_{a_1}, K_{b_1})(K_{a_2}, K_{b_2}) \cdots (K_{a_n}, K_{b_n}) \]

Choose a permutation \( \eta \) so that

\[ \eta^{-1} g \eta = (K_{a_1}, K_{b_1})(K_{a_2}, K_{b_2}) \cdots (K_{a_n}, K_{b_n}) \]

Hence,

\[ \eta^{-1} g \eta = \begin{bmatrix}
\sigma_1 & 0 \\
0 & \sigma_2 \\
\vdots & \ddots \\
0 & \ldots & \sigma_n
\end{bmatrix}, \text{ where } \sigma_i = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \]

The Log of a Unitary Matrix

Let \( U \) be an arbitrary finite \( r \times r \) unitary matrix.

Then eigenvalues of \( U \) all lie on the unit circle in the complex plane.

Moreover, there exists a unitary matrix \( W \) which diagonalizes \( U \), i.e., there exists a unitary matrix \( W \) such that

\[ W^* W = \begin{bmatrix}
\Delta & 0 \\
0 & \Delta
\end{bmatrix} \]

where \( e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n} \) are the eigenvalues of \( U \).

\[ \ln(\sigma_i) = \frac{i\pi}{2} (2s + 1)(\sigma_s - \sigma_i), \quad s \in \mathbb{Z} \]

For simplicity, we choose the branch \( s = 0 \).

\[ H_s = -i \eta \ln(\eta^{-1} g \eta) \eta^{-1} \]

The Log of a Unitary Matrix

Then

\[ \ln(U) = W^{-1} \Delta \left( \ln(e^{i\theta_1}), \ln(e^{i\theta_2}), \ldots, \ln(e^{i\theta_n}) \right) W \]

Since \( \ln(e^{i\theta_j}) = i\theta_j + 2\pi in_j \), where \( n_j \in \mathbb{Z} \) is an arbitrary integer, we have

\[ \ln(U) = iW^{-1} \Delta \left( \theta_1 + 2\pi n_1, \theta_2 + 2\pi n_2, \ldots, \theta_n + 2\pi n_n \right) W \]

where \( n_1, n_2, \ldots, n_n \in \mathbb{Z} \)
The Log of a Unitary Matrix

Since \( e^A = \sum_{m=0}^{\infty} A^m / (m!) \), we have

\[
e^{iA(t)} = e^{W^{-1}A\Delta(t)W} = W^{-1}\Delta(t)W = W^{-1}\Delta(e^{it}\theta_1, e^{it}\theta_2, \ldots, e^{it}\theta_n)W
\]

Since \( e^{i\Delta(t)} \), we have

Hamiltonians for \( A(n) \)

Using the Hamiltonian for the Reidemeister 2 move

\[
g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

and the initial state

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

we have that the solution to Schrödinger’s equation for time \( t \) is

\[
\begin{pmatrix} e^{(\Delta(t/2) \theta_1)} \\ e^{(\Delta(t/2) \theta_2)} \\ e^{(\Delta(t/2) \theta_n)} \end{pmatrix}
\]

Observables which are Quantum Knot Invariants

Question. What do we mean by a physically observable knot invariant?

Let \( (\mathcal{K}^{(n)}, A(n)) \) be a quantum knot system. Then a quantum observable \( \Omega \) is a Hermitian operator on the Hilbert space \( \mathcal{K}^{(n)} \).

Observable Q. Knot Invariants

Question. But which observables \( \Omega \) are actually knot invariants?

Def. An observable \( \Omega \) is an invariant of quantum knots provided \( U \Omega U^{-1} = \Omega \) for all \( U \in A(n) \).

Observable Q. Knot Invariants

Question. But how do we find quantum knot invariant observables?

Theorem. Let \( (\mathcal{K}^{(n)}, A(n)) \) be a quantum knot system, and let

\[
\mathcal{K}^{(n)} = \bigoplus_{\ell} W_{\ell}
\]

be a decomposition of the representation \( A(n) \times \mathcal{K}^{(n)} \to \mathcal{K}^{(n)} \) into irreducible representations.

Then, for each \( \ell \), the projection operator \( P_{\ell} \) for the subspace \( W_{\ell} \) is a quantum knot observable.
Theorem. Let \( \mathcal{K}^{(n)} \), \( A(n) \) be a quantum knot system, and let \( \Omega \) be an observable on \( \mathcal{K}^{(n)} \). Let \( \text{St} (\Omega) \) be the stabilizer subgroup for \( \Omega \), i.e.,
\[
\text{St} (\Omega) = \left\{ U \in A(n) : U\Omega U^{-1} = \Omega \right\}
\]
Then the observable
\[
\sum_{U \in A(n) / \text{St} (\Omega)} U\Omega U^{-1}
\]
is a quantum knot invariant, where the above sum is over a complete set of coset representatives of \( \text{St} (\Omega) \) in \( A(n) \).

Observable Q. Knot Invariants

• Orbit Projectors: For each mosaic knot \( K \), the observable
\[
P_{U(a,k)} = \sum_{a \in K^{(n)}} |K\rangle \langle K'|
\]
is a quantum knot \( n \)-invariant.

• Let \( I : K \rightarrow C \) be a knot invariant. Then
\[
\bar{I} = \sum_{a \in K^{(n)}} I(K) |K\rangle \langle K|
\]
is an observable which is a quantum knot \( n \)-invariant.

The following is an example of a quantum knot invariant observable:
\[
\Omega = |+\rangle \langle +| + |0\rangle \langle 0|
\]

Quantum \( \neq \) Quantum

A Quantum Knot Invariant \( \neq \) Quantum Invariant as is usually defined

Next Objective

We would like to find an equivalent definition of quantum knots that is more directly related to the spatial configurations of knots in 3-space.

Rationale: Mosaics are based on knot projections, and hence, only indirectly on the actual knot. Moreover, PI & Reidemeister moves are moves on knot projections, and hence, also only indirectly associated with the actual spatial configuration of knots.
Can we find an alternative approach to knot theory?

Before we ask this question, we need to find the answer to a more fundamental question:

**How does a dog wag its tail?**

My best friend Tazi knew the answer.

Tazi = Tasmanian Tiger

- She would **wiggle** her tail, just as a creature would squirm on a flat planar surface.
- She would **wag** her tail in a twisting corkscrew motion.
- Her tail would also stretch or contract when an impolite child would **tug** on it.

**Key Intuitive Idea**

A curve in 3-space has 3 local (i.e., infinitesimal) degrees of freedom.

- **Wiggle**: A curvature move
- **Wag**: A torsion move
- **Tug**: A metric move

Can we take this idea and use it to create a useable well-defined set of moves which will replace the Reidemeister moves?
Can we find an alternate set of knot moves that is much more “physics friendly” ???

For Mechanical Engineers, there are two types of knot theory.

**Extensible Knot Theory**, where moves that change the length of the knot are permitted. 

**Inextensible Knot Theory**, where moves that change the length of the knot are not permitted.

Wiggles and wags can be replaced by sequences of tugs.

Hence, for extensible knot theory, there is only one move, i.e.,

A tug

3-Bar Prismatic Linkage

which is the same as Reidemeister’s original triangle move.

We would like to discuss both

- **Extensible** (a.k.a., topological) knot theory, and
- **Inextensible** Knot Theory.

But because of time constraints, we will focus mainly on extensible knot theory.
The Tangle is a device that moves only by wagging.

The Bendangle is a device that moves only by wiggling. (Patent pending)

The Universal Bendangle is a device that moves only by wiggling and wagging. (Patent pending)

Quantum Knots & Lattices

The Cubic Honeycomb

A Scaffolding for 3-Space

For each non-negative integer $\ell$, let $A_\ell$ denote the 3-D lattice of points

$$A_\ell = \left\{ \left( \frac{m_1}{2^\ell}, \frac{m_2}{2^\ell}, \frac{m_3}{2^\ell} \right) : m_1, m_2, m_3 \in \mathbb{Z} \right\}$$

lying in Euclidean 3-space $\mathbb{R}^3$.

This lattice determines a tiling of $\mathbb{R}^3$ by $2^{-\ell} \times 2^{-\ell} \times 2^{-\ell}$ cubes, called the cubic honeycomb of $\mathbb{R}^3$ (of order $\ell$)
The Cubic Honeycomb

We think of this honeycomb as a cell complex $G$ for $\mathbb{R}^3$ consisting of:

- Vertices $a \in \mathcal{A}$
- Edges $E$
- Faces $F$
- Cubes $B$

All cells of positive dimension are open cells.

Lattice Knots

Definition. A lattice graph $G$ (of order $\ell$) is a finite subset of edges (together with their endpoints) of the honeycomb $G$.

Definition. A lattice knot $K$ (of order $\ell$) is a lattice 2-valent graph (of order $\ell$). Let $\mathcal{K}^{\ell}$ denote the set of all lattice knots of order $\ell$.

Lattice Knots

Lattice Trefoil

Lattice Hopf Link

Necessary Infrastructure

Orientation of 3-Space

We define an orientation of $\mathbb{R}^3$ by selecting a right handed frame

at the origin $O = (0,0,0)$ properly aligned with the edges of the honeycomb, and by parallel transporting it to each vertex $a \in \mathcal{A}$.

We refer to this frame as the preferred frame.
A vertex \( a \) of a cube \( B \) is called a preferred vertex of \( B \) if the first octant of the preferred frame at \( a \) contains the cube \( B \).

Since \( B \) is uniquely determined by its preferred vertex, we use the notation

\[
B = B^{(t)}(a)
\]
The Left and Right Permutations

Define the left and right permutations as:

\( L : \{1,2,3\} \rightarrow \{1,2,3\} \)

\[
\begin{align*}
1 & \mapsto 2 \\
2 & \mapsto 3 \\
3 & \mapsto 1
\end{align*}
\]

\( R : \{1,2,3\} \rightarrow \{1,2,3\} \)

\[
\begin{align*}
1 & \mapsto 3 \\
2 & \mapsto 1 \\
3 & \mapsto 2
\end{align*}
\]

Ergo, \( e_r = e_x \times e_y \), \( e_L = e_y \times e_z \), \( e_R = e_x \times e_z \).

Drawing Conventions

When drawn in isolation, \( F_p(a) \) is always drawn with preferred vertex \( a \) in the upper left hand corner, and with \( e_r(a) \) pointing out of the page.

Preferred Vertex

Invisible preferred frame

\( e_r(a) \) points out of the page toward the reader.

Vertex Translation

Let \( a \) be a vertex in the lattice \( \mathcal{L} \):

\[
\begin{align*}
a^p & = a + 2^t e_r \\
a^- & = a - 2^t e_r \\
a^+ & = a + 3 \cdot 2^t e_r
\end{align*}
\]

So for example,

\[
a^{+X3} = a + 2 \cdot 2^t e_x - 5 \cdot 2^t e_y + 2^t e_z
\]

Lattice knot moves

Definition. A lattice knot move \( \mu \) (of order \( t \)) is a bijection:

\[
\mu : \mathcal{K}(t) \rightarrow \mathcal{K}(t)
\]

The move \( \mu \) is said to be local if there exists a closed cube \( B(\mu)(a) \) in the lattice such that:

\[
\mu|_{B(\mu)(a)} = \text{id}|_{B(\mu)(a)}
\]

for all \( K \in \mathcal{K}(t) \).
We will now define the lattice knot moves tug, wiggle, and wag.

**Tugs**

\[ L_1^{(t)}(a, p, \square) = \square^{(t)}(a, p) \]

This is a local move on face \( F_p^{(t)}(a) \)

For each cube, 4 Tugs for each preferred face

Tugs are extensible local moves

**Wags**

\[ L_1^{(w)}(a, 1, \square) = \square^{(w)}(a, 1) \]

Wags are inextensible local moves
The Left and Right Permutations

Please recall that the left and right permutations \( \underline{\ } \) and \( \underline{\ } \) are defined as

\[
L : \{1,2,3\} \rightarrow \{1,2,3\}
\]

\[
L : \{1,2,3\} \rightarrow \{1,2,3\}
\]

\[
\begin{align*}
1 & \mapsto 2 \\
2 & \mapsto 3 \\
3 & \mapsto 1
\end{align*}
\]

\[
\begin{align*}
1 & \mapsto 3 \\
2 & \mapsto 1 \\
3 & \mapsto 2
\end{align*}
\]

\[
\begin{align*}
e_p &= e_L \times e_{rL} \\
e_{Lr} &= e_L \times e_{rL}
\end{align*}
\]

Preferred edges & Faces

Wags

For each cube, there are 4 wags for each of its 3 preferred faces.

Hence, there are 12 wags per cube.

Wags are inextensible local moves

4 Wags for face 1

An Example of a Wiggle Move
The Ambient Groups

Tug, Wiggle, & Wag are Permutations

For each \( \ell \geq 0 \), each of the above moves, Tug, Wiggle, & Wag, is a permutation (bijection) on the set \( \mathcal{K}^{(\ell)} \) of all lattice knots of order \( \ell \).

In fact, each of the above local moves, as a permutation, is the product of disjoint transpositions.

Definition. The ambient group \( \Lambda_\ell \) is the group generated by tugs, wiggles, and wags of order \( \ell \).

Definition. The inextensible ambient group \( \overline{\Lambda}_\ell \) is the group generated only by wiggles and wags of order \( \ell \).

Tugs, wiggles, and wags are a set of involutions that generate the above groups.

What Is the Ambient Group?

We create a representation of the Ambient Group onto a group of Conditional OP Autohomeomorphisms of \( \mathbb{R}^3 \).

What is the ambient group???

Observation: Wiggle, wag, and tug are symbolic conditional moves, as are the Reidemeister moves.

For example, the tug

\[
\begin{align*}
\square^{(\ell)}(a,p) &= \square^{(\ell)}(a,p) \bigcup \square^{(\ell)}(a,p) \\
&= \square^{(\ell)}(a,p) \bigcup \square^{(\ell)}(a,p) \\
\square^{(\ell)}(a,p) &= \square^{(\ell)}(a,p) \\
&= \square^{(\ell)}(a,p) \\
\square^{(\ell)}(a,p) &= \square^{(\ell)}(a,p) \\
&= \square^{(\ell)}(a,p)
\end{align*}
\]
What is the ambient group ???

Observation: Each is a symbolic representation of an authentic conditional move, i.e., a conditional orientation preserving (OP) auto-homeomorphism of $\mathbb{R}^3$.

Moreover, each involved (OP) auto-homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ is local, i.e., there exists a 3-ball $D$ such that

$$h|_{\mathbb{R}^3 \backslash D} = \text{id} : \mathbb{R}^3 \to \mathbb{R}^3$$

Let $\mathcal{L}$ be a family of knots in $\mathbb{R}^3$.

Def. A local authentic conditional (LAC) move on a family of knots $\mathcal{L}$ is a map

$$\Phi : \mathcal{L} \to \mathcal{L}(\mathbb{R}^3)$$

such that

$$\Phi(k) \in \mathcal{L} \forall k \in \mathcal{L}$$

Let $\mathcal{L}(\mathbb{R}^3)$ be the space of all LAC moves for the family $\mathcal{L}$.

Proposition. $\mathcal{L}(\mathbb{R}^3)$ is a monoid.

In the paper "Quantum Knots and Lattices," we construct a faithful representation

$$\Gamma : \Lambda \to \mathcal{L}(\mathbb{R}^3)$$

into a subgroup of the monoid $\mathcal{L}(\mathbb{R}^3)$ by mapping each generator wiggle, wag, and tug onto a local conditional OP auto-homeomorphism of $\mathbb{R}^3$.

What is the ambient group ???

Let $\mathcal{L}(\mathbb{R}^3)$ be the group of local OP auto-homeomorphisms of $\mathbb{R}^3$.

For example:

$\mathcal{L}(\mathbb{R}^3)$ The family of lattice knots

$\mathcal{S}$ The family of finitely piecewise smooth (FPWS) knots in $\mathbb{R}^3$.

Refinement
The refinement injection

**Def.** We define the refinement injection \( \Upsilon: \mathcal{R}(\ell) \rightarrow \mathcal{R}(\ell+1) \) from lattice knots of order \( \ell \) to lattice knots of order \( \ell+1 \) as

\[
\Upsilon: \mathcal{R}(\ell) \rightarrow \mathcal{R}(\ell+1), \quad K \mapsto \bigcup_{a \in A_{\ell}} \bigcup_{p=1}^{3} \bigcup_{e_f \in \text{edges}} \{E_f(a), E_f(a')\}
\]

An example:

\( \Upsilon(\mathcal{R}(\ell)) \quad \rightarrow \quad \Upsilon(\mathcal{R}(\ell+1)) \)

Conjectured Refinement Monomorphism

We conjecture the existence of a refinement monomorphism \( \Upsilon: \Lambda \rightarrow \Lambda_{\ell+1} \) which preserves the action

\[
\Lambda \times \mathcal{R}(\ell) \rightarrow \mathcal{R}(\ell+1), \quad (g, K) \mapsto gK
\]

i.e., with the property

\[
\Upsilon(g)\Upsilon(K) = \Upsilon(gK)
\]

In fact, we have a construction which we believe is such a monomorphism.

The Refinement Morphism \( \Upsilon: \Lambda \rightarrow \Lambda_{\ell+1} \)

\( \Upsilon(\square(a,1)) = a^\ell b = b^{-\ell}ab \)

\( \square(a^2,1) \times \square(a^2,1) \times \square(a^2,1) \times \square(a,1) \times \square(a,1) \)

Knot Type
**Lattice Knot Type**

Two lattice knots $K_1$ and $K_2$ in $\mathbb{R}^3$ are of the same $\ell$-type, written $K_1 \approx_{\ell} K_2$, provided there is an element $g \in \Lambda_\ell$ such that $gK_1 = K_2$.

They are of the same knot type, written $K_1 \approx K_2$, provided there is a non-negative integer $m$ such that $\mathcal{R}^m K_1 \approx \mathcal{R}^m K_2$.

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**Inextensible Lattice Knot Type**

Two lattice knots $K_1$ and $K_2$ in $\mathbb{R}^3$ are of the same inextensible $\ell$-type, written $K_1 \approx_{\ell} K_2$, provided there is an element $g \in \tilde{\Lambda}_\ell$ such that $gK_1 = K_2$.

They are of the same inextensible knot type, written $K_1 \approx K_2$, provided there is a non-negative integer $m$ such that $\mathcal{R}^m K_1 \approx \mathcal{R}^m K_2$.

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**n-Bounded Lattices, Lattice knots, and Ambient Groups**

In preparation for creating a definition of physically implementable quantum knot systems, we need to work with finite mathematical objects.

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**n-Bounded Lattices**

Let $\ell$ and $n$ be be non-negative integers. We define the $n$-bounded lattice of order $\ell$ as $\mathcal{L}_{\ell,n} = \{ a \in \mathcal{L}_\ell : |a| \leq n \}$, where $|a| = \max(|a_i|)$.

We also have $\mathcal{C}_{\ell,n}$, the corresponding cell complex, and $\mathcal{C}_{\ell,n}^j$, the corresponding $j$-skeleton.

---

**n-Bounded Lattice Knots & Ambient Groups**

The set of $n$-bounded lattice knots of order $\ell$ is $\mathcal{R}^{(\ell,n)} = \mathcal{R}^{(n)} \cap \mathcal{L}_{\ell,n}$.

The Ambient group of order $(\ell,n)$ is $\Lambda_{\ell,n} = \Lambda_{\ell} \mid_{\mathcal{L}_{\ell,n}}$.

The Inextensible Ambient group of order $(\ell,n)$ is $\tilde{\Lambda}_{\ell,n} = \tilde{\Lambda}_{\ell} \mid_{\mathcal{L}_{\ell,n}}$. 
n-Bounded Lattice Knots & Ambient Groups

We also have the injection
\( \iota : \mathcal{R}^{(\ell,n)} \to \mathcal{R}^{(\ell,n+1)} \)
and the monomorphisms
\( \iota : \Lambda^{(\ell,n)} \to \Lambda^{(\ell,n+1)} \) and \( \iota : \hat{\Lambda}^{(\ell,n)} \to \hat{\Lambda}^{(\ell,n+1)} \)

We thus have a nested sequence of lattice knot systems
\[ \mathcal{R}^{(\ell,1)} \to \mathcal{R}^{(\ell,2)} \to \cdots \to \mathcal{R}^{(\ell,n)} \to \cdots \]

Quantum Knots & Quantum Knot Systems

For Q.M. systems, we need an underlying Hilbert space. So we define:

The Hilbert space \( \mathcal{G}^{(\ell,n)} \) of quantum knots

Recall that \( \mathcal{R}^{(\ell,n)} \) is the set of all lattice knots of order \( (\ell,n) \).

For Q.M. systems, we need an underlying Hilbert space. So we define:

The Hilbert space \( \mathcal{G}^{(\ell,n)} \) of quantum knots

is the Hilbert space with the set \( \mathcal{R}^{(\ell,n)} \) of lattice knots of order \( (\ell,n) \) as its orthonormal basis, i.e., with orthonormal

\[ \{ [K] : K \in \mathcal{R}^{(\ell,n)} \} \]

n-Bounded Lattice Knot Type

Two lattice knots \( K_1 \) and \( K_2 \) in \( \mathcal{R}^{(\ell,n)} \) are said to be of the same lattice knot type, written \( K_1 \sim K_2 \), provided there is an element \( g \in \Lambda^{(\ell,n)} \) such that
\[ gK_1 = K_2 \]

They are of the same lattice knot type, written \( K_1 \sim K_2 \), provided there are non-negative integers \( \ell' \) and \( n' \) such that
\[ \ell' \sim \ell \quad n' \sim n \]

In like manner for the inextensible ambient group \( \overline{\Lambda}^{(\ell,n)} \), we can define

\[ K_1 \sim K_2 \quad \text{and} \quad K_1 = K_2 \]
Quantum Knots

E Edge Coloring Space

\( E = 2\text{-D Hilbert space with orthonormal basis} \)

Non-Edge
Non-Edge

“Hollow” Gray

Existent Edge

“Solid” Red

\( |0\rangle = \begin{array}{c}
\vdots \\
|1\rangle = \end{array} \)

\( G^{(\alpha)} \) Hilbert Space of Lattice Graphs in \( L_{\alpha} \)

\( G^{(\alpha)} = \bigotimes_{e \in \text{Edges}(G_\alpha)} E \)

Orthonormal basis is:

\[ \left\{ |G\rangle : G \text{ lattice graph in } L_{\alpha} \right\} \]

Quantum Knots

An Example of a Quantum Knot

\[ |K\rangle = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \sqrt{2} \]

The Ambient Group \( \Lambda_{\alpha} \) as a Unitary Group

We identify each element \( g \in \Lambda_{\alpha} \) with the linear transformation defined by

\[ \mathcal{K}^{(\alpha)} \rightarrow \mathcal{K}^{(\alpha)} \]

\[ |K\rangle \mapsto |gK\rangle \]

Since each element \( g \in \Lambda_{\alpha} \) is a permutation, it is a linear transformation that simply permutes basis elements.

Hence, under this identification, the ambient group \( \Lambda_{\alpha} \) becomes a discrete group of unitary transforms on the Hilbert space \( \mathcal{K}^{(\alpha)} \).
An Example of the $\Lambda_{\ell, n}$ Group Action

$$K = \begin{pmatrix} \text{Vertex } e^1 \\ \text{Face } F_{10}(e^2) \end{pmatrix}$$

Wiggle $a^\ell(a^3, 3)$

$$\square a^\ell(a^3, 3)(|K\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} |K\rangle \\ |K'\rangle \end{pmatrix}$$

The Quantum Knot System $(\mathcal{K}^{(\ell, n)}_{\Lambda_{\ell, n}})$

**Def.** A quantum knot system $(\mathcal{K}^{(\ell, n)}_{\Lambda_{\ell, n}})$ is a quantum system having $\mathcal{K}^{(\ell, n)}_{\Lambda_{\ell, n}}$ as its state space, and having the Ambient group $\Lambda_{\ell, n}$ as its set of accessible unitary transformations.

The states of quantum system $(\mathcal{K}^{(\ell, n)}_{\Lambda_{\ell, n}})$ are quantum knots. The elements of the ambient group $\Lambda_{\ell, n}$ are quantum moves.

$$\mathcal{K}^{(\ell, n)}_{\Lambda_{\ell, n}} \to \mathcal{K}^{(\ell, n)}_{\Lambda_{\ell, n}} \to \mathcal{K}^{(\ell, n+1)}_{\Lambda_{\ell, n}} \to \ldots$$

Physically Implementable

Physically Implementable

Physically Implementable

Choosing integers $\ell$ and $n$ is analogous to choosing respectively the thickness and the length of the rope. The smaller the thickness and the longer the rope, the more knots that can be tied.

The parameters (wiggle, wag, & tug) of the ambient group $\Lambda_{\ell, n}$ are the “knobs” one turns to spatially manipulate the quantum knot.

Two Quantum Knots of the Same Knot Type

$$K_1 = \begin{pmatrix} \begin{pmatrix} \text{Vertex } e^1 \\ \text{Face } F_{10}(e^2) \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} \text{Vertex } e^1 \\ \text{Face } F_{10}(e^2) \end{pmatrix} \end{pmatrix}$$

Wiggle $a^\ell(a^3, 3)$

$$\square a^\ell(a^3, 3)(|K\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} |K\rangle \\ |K'\rangle \end{pmatrix}$$

Two Quantum Knots NOT of the Same Knot Type

$$K_2 = \begin{pmatrix} \begin{pmatrix} \text{Vertex } e^1 \\ \text{Face } F_{10}(e^2) \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} \text{Vertex } e^1 \\ \text{Face } F_{10}(e^2) \end{pmatrix} \end{pmatrix}$$

Quantum Knot Type

**Def.** Two quantum knots $|K_1\rangle$ and $|K_2\rangle$ are of the same knot $\ell, n$-type, written $K_1 \sim K_2$ provided there is an element $g \in \Lambda_{\ell, n}$ s.t.

$$g |K_1\rangle = |K_2\rangle$$

They are of the same knot type, written $K_1 \sim K_2$, provided there are integer $\ell', n' \geq 0$ such that

$$\mathcal{R}^{\ell'} |K_1\rangle \underset{\ell' \geq 0}{\sim} \mathcal{R}^{n'} |K_2\rangle$$

Physically Implementable

Physically Implementable

Physically Implementable
Hamiltonians of the Generators of the Ambient Group

Hamiltonians for $A(n)$

Also, let $\sigma_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and note that

$$\ln(\sigma_s) = \frac{i\pi}{2} (2s+1) (\sigma_s - \sigma_i), \quad s \in \mathbb{Z}$$

For simplicity, we always choose the branch $s = 0$.

$$H_e = -i\eta \ln(\eta^{-1} g \eta) \eta^{-1}$$

$$= \frac{\pi}{2} \eta \begin{pmatrix} I_s \otimes (\sigma_i - \sigma_j) & 0 \\ 0 & 0 \end{pmatrix} \eta^{-1}$$

The Log of a Unitary Matrix

Let $U$ be an arbitrary finite $r \times r$ unitary matrix.

Then eigenvalues of $U$ all lie on the unit circle in the complex plane.

Moreover, there exists a unitary matrix $W$ which diagonalizes $U$, i.e., there exists a unitary matrix $W$ such that

$$W U W^{-1} = \Delta(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r})$$

where $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}$ are the eigenvalues of $U$. 

The Log of a Unitary Matrix

Then

$$\ln(U) = W^{-1} \Delta(\ln(e^{i\theta_1}), \ln(e^{i\theta_2}), \ldots, \ln(e^{i\theta_r})) W$$

Since $\ln(e^{i\theta}) = i\theta_j + 2\pi j n_j$, where $n_j \in \mathbb{Z}$ is an arbitrary integer, we have

$$\ln(U) = i W^{-1} \Delta(\theta_1 + 2\pi n_1, \theta_2 + 2\pi n_2, \ldots, \theta_r + 2\pi n_r) W$$

where $n_1, n_2, \ldots, n_r \in \mathbb{Z}$.
Hamiltonians for $\Lambda_{f,a}$

Using the Hamiltonian for the wiggle move

$U(\alpha,3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and the initial state $|\psi\rangle$

we have that the solution to Schrödinger’s equation for time $t$ is

$e^{i\pi\alpha t} |\psi\rangle = \sin(\pi t) |\psi\rangle - \cos(\pi t) |\psi\rangle$

and the initial state $|\psi\rangle$.

We have that the solution to Schrödinger’s equation for time $t$ is

$|\psi(t)\rangle = e^{-iE_{f,a} t} |\psi\rangle$

Initial state

Observables

which are

Quantum Knot Invariants

Question. What do we mean by a physically observable knot invariant?

Let $(K^{f,a},\Lambda_{f,a})$ be a quantum knot system. Then a quantum observable $O$ is a Hermitian operator on the Hilbert space $K^{f,a}$.

Observables

Quantum Knot Invariants

Question. But which observables $O$ are actually knot invariants?

Definition. An observable $O$ is an invariant of quantum knots provided $UOU^{-1} = O$ for all $U \in \Lambda_{f,a}$.

Observables

Quantum Knot Invariants

Theorem. Let $(K^{f,a},\Lambda_{f,a})$ be a quantum knot system, and let $O$ be an observable on $K^{f,a}$. Let $St(O)$ be the stabilizer subgroup for $O$, i.e.,

$St(O) = \{ U \in A(n) : UOU^{-1} = O \}$

Then the observable

$\sum_{U \in St(O)} O UOU^{-1}$

is a quantum knot invariant, where the above sum is over a complete set of coset representatives of $St(O)$ in $\Lambda_{f,a}$.
In $\mathcal{K}(n)$, the following is an example of an inextensible quantum knot invariant observable:

$$\Omega = \sum_{a} \frac{\partial F_{p}^{(n)}(a)^{\alpha} \partial F_{p}^{(n)}(a)}{\partial F_{p}^{(n)}(a)} + \sum_{a} \frac{\partial F_{p}^{(n)}(a^{\alpha}) \partial F_{p}^{(n)}(a^{\alpha})}{\partial F_{p}^{(n)}(a^{\alpha})}$$

where $\partial F_{p}^{(n)}(a)$ denotes the boundary of the face $F_{p}^{(n)}(a)$.

Future Directions & Open Questions

• What is the structure of the ambient groups $\Lambda_{t}, \Lambda_{h}, \Lambda_{t_{1}}, \Lambda_{t_{2}}$, and their direct limits? Can one find a presentation of these groups? Are they Coxeter groups? Can one find a presentation of the direct limits of these groups?

• Exactly how are the lattice and the mosaic ambient groups related to one another?

• For each lattice knot $K$, let $V_{K}(t)$ denote its Jones polynomial. We define the Jones observable as

$$V_{K}^{(n)}(t) = \sum_{K \in \mathcal{K}(n)} V_{K}(t) |K\rangle \langle K|$$

How is the Jones observable related to the Aharonov, Jones, Landau algorithm? Can it be used to create a different quantum algorithm for the Jones polynomial?

• Unlike classical knots, quantum knots can exhibit the non-classical behavior of quantum superposition and quantum entanglement. Are quantum and topological entanglement related to one another? If so, how?

• What is gained by extending the definition of quantum knot observables to POVMs?

• What is gained by extending the definition of quantum knot observables to mixed ensembles?
Future Directions & Open Questions

Def. We define the lattice number of a knot $K$ as the smallest integer $n$ for which $K$ is representable as a lattice knot of order $(\ell=0,n)$.

How does one compute the lattice number of a knot? How does one find a quantum observable for the lattice number?

How is the lattice number related to the mosaic number of a knot?

Quantum Knot Tomography: Given many copies of the same quantum knot, find the most efficient set of measurements that will determine the quantum knot to a chosen tolerance $\varepsilon > 0$.

Quantum Braids: Use lattices to define quantum braids. How are such quantum braids related to the work of Freedman, Kitaev, et al on anyons and topological quantum computing?

Tug, Wiggle, & Wag are “physics friendly”

Reason: From these moves, we can create by taking the limit as $\ell \to \infty$

- Variational derivatives w.r.t. moves, e.g.,
  \[ \frac{\delta F[x]}{\delta \Box(a,f)} \]

- Infinitesimal moves, e.g.,
  \[ \Box(x)^{\frac{2}{x^2+y^2}} \]

- Move differential forms, e.g.,
  \[ \Box(x)^{x+y} \]

- Multiplicative integrals of diff. forms, e.g.,
  \[ \int_{x=0}^{1} \int_{y=0}^{1} \Box(x_1,x_2,0)^{x+y} \]

Variational Derivatives w.r.t. moves

\[
\frac{\delta F[x]}{\delta \Box(a,f)} \quad \frac{\delta F[x]}{\delta \Box(a,f)} \quad \frac{\delta F[x]}{\delta \Box(a,f)}
\]

The Preferred Vertex (PV) Map $[-]$

Since the half closed cubes form a partition of 3-space $\mathbb{R}^3$, we have the preferred vertex map

\[ PV^{(t)}: \mathbb{R}^3 \to 4 \equiv \left\{ \left( m_1, m_2, m_3 : m_1, m_2, m_3 \in \mathbb{Z} \right) \right\} \]
A finitely piecewise smooth (FPS) knot is a knot $x$ in 3-space $\mathbb{R}^3$ that consists of finitely many piecewise smooth ($C^\infty$) segments with no two consecutive segments meeting in a tangential cusp.

Let $F_{FPS}$ be the family of FPS knots.

Let $F : F_{FPS} \to \mathbb{R}$ be a real valued functional on $F_{FPS}$.

Total Tug, Total Wiggle, & Total Wag

Total Tug:
$$\int a(a,f) = \int a(a,f) \int a(a,f) \int a(a,f) \int a(a,f)$$

Total Wiggle:
$$\int a(a,f) = \int a(a,f) \int a(a,f)$$

Total Wag:
$$\int a(a,f) = \int a(a,f) \int a(a,f) \int a(a,f) \int a(a,f)$$

Variational Derivatives

Conjecture: A functional $F : F_{FPS} \to \mathbb{R}$ is a knot invariant if all its variational derivatives exist and are zero.

Total Tug, Total Wiggle, & Total Wag

Total Tug:
$$\int a(a,f) = \int a(a,f) \int a(a,f) \int a(a,f) \int a(a,f)$$

Total Wiggle:
$$\int a(a,f) = \int a(a,f) \int a(a,f)$$

Total Wag:
$$\int a(a,f) = \int a(a,f) \int a(a,f) \int a(a,f) \int a(a,f)$$

Variational Derivatives w.r.t. moves

$$\frac{\delta F[x]}{\delta \int a(a,f)} = \lim_{\epsilon \to 0} F \left[ \int a(a,f) \mathcal{P}V(a,f)(x) \right] - F[x]$$

$$\frac{\delta F[x]}{\delta \int a(a,f)} = \lim_{\epsilon \to 0} F \left[ \int a(a,f) \mathcal{P}V(a,f)(x) \right] - F[x]$$

$$\frac{\delta F[x]}{\delta \int a(a,f)} = \lim_{\epsilon \to 0} F \left[ \int a(a,f) \mathcal{P}V(a,f)(x) \right] - F[x]$$

UMBC Quantum Knots Research Lab

We at UMBC are very proud of our new state of the art Quantum Knots Research Laboratory.

We have just purchased some of the latest and most advanced equipment in quantum knots research !!!