Quantum Knots

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Two papers on Quantum Knots can be found in this book.

Throughout this talk:

“Knot” means either a knot or a link

This talk is based on the papers:


Knots Naturally Arise in the Quantum World as Dynamical Processes

Examples of dynamical knots in quantum physics:

**Knotted vortices**
- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

Reason for current intense interest:

Topology Is a Natural Obstruction to Decoherence

Thinking Outside the Box

Quantum Mechanics

is a tool for exploring

Knot Theory

This talk was also motivated by:


All the papers on Quantum Knots can be found on Quant-Ph and on the website.

www.csee.umbc.edu/~lomonaco

PowerPoint slides can be found at:

www.csee.umbc.edu/~lomonaco/Lectures.html
**Quantum Topology ≠ Quantum Physics**

- The objective of this work is to do topology in such a way that it is intimately related to quantum physics.
- The ultimate objective is to create and to investigate mathematical objects that can be physically implemented in a quantum physics lab.

**Objectives**

- We seek to create a quantum system that simulates a closed knotted physical piece of rope.
- We seek to define a quantum knot in such a way as to represent the state of the knotted rope, i.e., the particular spatial configuration of the knot tied in the rope.
- We also seek to model the ways of moving the rope around (without cutting the rope, and without letting it pass through itself.)

**Rules of the Game**

Find a mathematical definition of a quantum knot that is

- Physically meaningful, i.e., physically implementable, and
- Simple enough to be workable and useable.

**Aspirations**

We would hope that this definition will be useful in modeling and predicting the behavior of knotted vortices that actually occur in quantum physics such as

- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

**At least 3 equiv ways to create Quantum Knots**

1) Mosaic Construction:
   - Advantages:
     - Easiest least technical approach
   - Disadvantages:
     - Geometry of 3-space not transparent
2) Lattice Construction:
   - Advantages:
     - Geometry of 3-space transparent
     - Moves become variational derivatives
     - Fits seamlessly into physics
   - Disadvantages:
     - Fraught with technicalities

**Lattice Approach**

With fear and trepidation, we must abandon the Reidemeister moves, and replace them with the truly 3-D moves: Wiggle, Wag, & Tug so named because these moves mimic how a dog moves its tail.

Curvature move
Torsion move
Metric move
At least 3 equiv ways to create Quantum Knots

3) Smooth curves in $\mathbb{R}^3$ construction:
   • Advantages:
     - No longer working with PL knots
     - All the advantages of lattice approach
   • Disadvantages:
     - Many technical obstacles

Outline

Knot Theory = Formal Rewriting System

Formal Rewriting System = Group Representation

Quantum Mechanics = Group Representation Theory

Mosaic Knots

Part 1

Mosaic Knots

Mosaic Tiles

Let $T^{(n)}$ denote the following set of 11 symbols, called mosaic (unoriented) tiles:

Please note that, up to rotation, there are exactly 5 tiles

Definition of an n-Mosaic

An n-mosaic is an $n \times n$ matrix of tiles, with rows and columns indexed $0, 1, \ldots, n-1$

An example of a 4-mosaic
**Tile Connection Points**

A connection point of a tile is a midpoint of an edge which is also the endpoint of a curve drawn on a tile. For example,

- 0 Connection Points
- 2 Connection Points
- 4 Connection Points

**Contiguous Tiles**

Two tiles in a mosaic are said to be contiguous if they lie immediately next to each other in either the same row or the same column.

**Suitably Connected Tiles**

A tile in a mosaic is said to be suitably connected if all its connection points touch the connection points of contiguous tiles. For example,

- Suitably Connected
- Not Suitably Connected

**Knot Mosaics**

A knot mosaic is a mosaic with all tiles suitably connected. For example,

- Non-Knot 4-Mosaic
- Knot 4-Mosaic

**Figure Eight Knot 5-Mosaic**

**Hopf Link 4-Mosaic**
**Borromean Rings 6-Mosaic**

**Notation**

\[ M^{(n)} = \text{Set of } n\text{-mosaics} \]

\[ K^{(n)} = \text{Subset of knot } n\text{-mosaics} \]

---

**Planar Isotopy Moves**

**Non-Deterministic Tiles**

We use the following tile symbols to denote one of two possible tiles:

For example, the tile \( \) denotes either

or

---

**11 Planar Isotopy (PI) Moves on Mosaics**

It is understood that each of the above moves depicts all moves obtained by rotating the 2x2 sub-mosaics by 0, 90, 180, or 270 degrees.

For example, \( \) represents each of the following 4 moves:
**Terminology: k-Submosaic Moves**

**Def.** A k-submosaic move on a mosaic M is a mosaic move that replaces one k-submosaic in M by another k-submosaic.

All of the PI moves are examples of 2-submosaic moves. I.e., each PI move replaces a 2-submosaic by another 2-submosaic.

For example, [image of a mosaic move]

**Planar Isotopy (PI) Moves on Mosaics**

Each of the PI 2-submosaic moves represents any one of the \((n-k+1)^2\) possible moves on an n-mosaic.

**Reidemeister Moves**

Each PI move acts as a local transformation on an n-mosaic whenever its conditions are met. If its conditions are not met, it acts as the identity transformation.

Ergo, each PI move is a permutation of the set of all knot n-mosaics \(K^{(n)}\).

In fact, each PI move, as a permutation, is a product of disjoint transpositions.

**Reidemeister (R) Moves on Mosaics**

**Reidemeister 1 Moves**

**Reidemeister 2 Moves**

**More Non-Deterministic Tiles**

We also use the following tile symbols to denote one of two possible tiles:

For example, the tile [image of a tile] denotes either [image of two possible tiles] or [image of another possible tile].
Synchronized Non-Deterministic Tiles

Nondeterministic tiles labeled by the same letter are synchronized:

\[
\begin{align*}
A & \iff A \\
\bar{A} & \iff \bar{A} \\
B & \iff B \\
\bar{B} & \iff \bar{B}
\end{align*}
\]

Reidemeister 3 (R3) Moves on Mosaics

Just like each PI move, each R move is a permutation of the set of all knot n-mosaics \( K^{(n)} \)

In fact, each R move, as a permutation, is a product of disjoint transpositions.

The Ambient Group \( A(n) \)

We define the ambient isotopy group \( A(n) \) as the subgroup of the group of all permutations of the set \( K^{(n)} \) generated by the all PI moves and all Reidemeister moves.

Knot Type

\[
\begin{align*}
\tau : M^{(n)} & \rightarrow M^{(n+1)} \\
M^{(n+1)}_{i,j} &= \begin{cases} 
M^{(n)}_{i,j} & \text{if } 0 \leq i, j < n \\
\text{otherwise} &
\end{cases}
\end{align*}
\]

The Mosaic Injection \( \iota : M^{(n)} \rightarrow M^{(n+1)} \)

We define the mosaic injection \( \iota : M^{(n)} \rightarrow M^{(n+1)} \)
**Mosaic Knot Type**

**Def.** Two $n$-mosaics $M$ and $M'$ are of the same **knot $n$-type**, written

$$M^n \sim M'^n$$

provided there exists an element of the ambient group $A(n)$ that transforms $M$ into $M'$.

Two $n$-mosaics $M$ and $M'$ are of the same **knot type** if there exists a non-negative integer $k$ such that

$$t^k M^n \sim t^k M'^n$$
Oriented Mosaics

There are 29 oriented tiles, and 9 tiles up to rotation. Rotationally equivalent tiles have been grouped together.

Oriented Mosaics and Oriented Knot Type

In like manner, we can use the following oriented tiles to construct oriented mosaics, oriented mosaic knots, and oriented knot type.

The Hilbert Space $\mathcal{M}^{(n)}$ of n-mosaics

Let $\mathcal{H}$ be the 11-dimensional Hilbert space with orthonormal basis labeled by the tiles.

We define the Hilbert space $\mathcal{M}^{(n)}$ of n-mosaics as

$$\mathcal{M}^{(n)} = \bigotimes_{i=1}^{n} \mathcal{H}$$

This is the Hilbert space with induced orthonormal basis

$$\{ \bigotimes_{i=1}^{n} |T_{k(i)}\rangle : 0 \leq k(i) < 11 \}$$

Identification via Row Major Order

Let $\mathcal{H}$ be the 11-dimensional Hilbert space with orthonormal basis labeled by the tiles.

For example, in the 3-mosaic Hilbert space $\mathcal{M}^{(3)}$, the basis ket $\bigotimes_{i=1}^{3} |T_{k(i)}\rangle$ is identified with the 3-mosaic labeled ket $T_2 T_5 T_1, T_5 T_2 T_1, T_2 T_1 T_5$.
The Hilbert Space $\mathcal{K}^{(n)}$ of Quantum Knots

The Hilbert space $\mathcal{K}^{(n)}$ of quantum knots is defined as the sub-Hilbert space of $\mathcal{M}^{(n)}$ spanned by all orthonormal basis elements labeled by knot $n$-mosaics.

Quantum Knots

We define the Hilbert space $\mathcal{M}^{(n)}$ of $n$-mosaics as

$$\mathcal{M}^{(n)} = \bigotimes_{\ell=0}^{n-1} \mathcal{K}$$

This is the Hilbert space with induced orthonormal basis

$$\{ \bigotimes_{\ell=0}^{n-1} |T_{\ell n}\rangle : 0 \leq \ell < n \}$$

We identify each basis element $\bigotimes_{\ell=0}^{n-1} |T_{\ell n}\rangle$ with the mosaic labeled ket $|M\rangle$ via the bijection $T_\ell \leftrightarrow M_{i,j}$ (Row major order)

where

$$i = \ell \, \lfloor \ell / n \rfloor$$

$$j = \ell - \lfloor \ell / n \rfloor$$

and $\ell = ni + j$

An Example of a Quantum Knot

$$|K\rangle = \frac{1}{\sqrt{2}} \left( |\rangle + \rangle \right)$$

The Ambient Group $A(n)$ as a Unitary Group

We identify each element $g \in A(n)$ with the linear transformation defined by

$$\mathcal{K}^{(n)} \rightarrow \mathcal{K}^{(n)}$$

$$|K\rangle \mapsto |gK\rangle$$

Since each element $g \in A(n)$ is a permutation, it is a linear transformation that simply permutes basis elements.

Hence, under this identification, the ambient group $A(n)$ becomes a discrete group of unitary transfs on the Hilbert space $\mathcal{K}^{(n)}$.

An Example of the $A(n)$ Group Action

$$|K\rangle = \frac{1}{\sqrt{2}} \left( \right)$$

$$R_1|K\rangle = \frac{1}{\sqrt{2}} \left( \right)$$

$$R_2|K\rangle = \frac{1}{\sqrt{2}} \left( \right)$$

The Quantum Knot System $\left(\mathcal{K}^{(n)}, A(n)\right)$

Def. A quantum knot system $\left(\mathcal{K}^{(n)}, A(n)\right)$ is a quantum system having $\mathcal{K}^{(n)}$ as its state space, and having the Ambient group $A(n)$ as its set of accessible unitary transformations.

The states of quantum system $\left(\mathcal{K}^{(n)}, A(n)\right)$ are quantum knots. The elements of the ambient group $A(n)$ are quantum moves.

$$\left(\mathcal{K}^{(n)}, A(1)\right) \rightarrow \cdots \rightarrow \left(\mathcal{K}^{(n)}, A(n)\right) \rightarrow \left(\mathcal{K}^{(n)}, A(n+1)\right) \rightarrow \cdots$$

Physically Implementable

Physically Implementable

Physically Implementable
The Quantum Knot System \((\mathcal{K}^n, A(n))\)

Choosing an integer \(n\) is analogous to choosing a length of rope. The longer the rope, the more knots that can be tied.

The parameters of the ambient group \(A(n)\) are the "knobs" one turns to spatially manipulate the quantum knot.

Quantum Knot Type

**Def.** Two quantum knots \(|K_1\rangle\) and \(|K_2\rangle\) are of the same knot type, written

\[ |K_1\rangle \sim |K_2\rangle, \]

provided there is an element \(g \in A(n)\) s.t.

\[ g|K_1\rangle = |K_2\rangle. \]

They are of the same knot type, written

\[ |K_1\rangle \sim |K_2\rangle, \]

provided there is an integer \(m \geq 0\) such that

\[ t^m |K_1\rangle \sim t^n |K_2\rangle. \]

Two Quantum Knots of the Same Knot Type

\[ |K\rangle = \frac{1}{\sqrt{2}} \left( |\bigcirc\rangle + |\square\rangle \right) \]

\[ R_2|K\rangle = \frac{1}{\sqrt{2}} \left( |\bigcirc\rangle + |\square\rangle \right) \]

Two Quantum Knots NOT of the Same Knot Type

\[ |K_1\rangle = \frac{1}{\sqrt{2}} \left( |\bigcirc\rangle + |\square\rangle \right) \]

\[ |K_2\rangle = \frac{1}{\sqrt{2}} \left( |\bigcirc\rangle + |\square\rangle \right) \]

Hamiltonians for \(A(n)\)

Each generator \(g \in A(n)\) is the product of disjoint transpositions, i.e.,

\[ g = (K_{a_1}K_{b_1})(K_{a_2}K_{b_2}) \cdots (K_{a_l}K_{b_l}) \]

Choose a permutation \(\eta\) so that

\[ \eta^{-1}g\eta = (K_1K_2)(K_3K_4) \cdots (K_lK_1) \]

Hence,

\[ \eta^{-1}g\eta = \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & 0 \\ 0 & \sigma_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_l \end{pmatrix}, \quad \text{where} \quad \sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
Also, let $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and note that

$$\ln(\sigma) = \frac{i \pi}{2} (2s+1)(\sigma_s - \sigma_1), \quad s \in \mathbb{Z}.$$ 

For simplicity, we always choose the branch $s = 0$.

$$H_g = -i \eta \ln(\eta^{-1} g \eta) \eta^{-1}$$

$$= \frac{\pi}{2} \eta \left( I_t \otimes (\sigma_0 - \sigma_1) \begin{pmatrix} 0 & \cos(2t) \\ \sin(2t) & 0 \end{pmatrix} \right) \eta^{-1}$$

Using the Hamiltonian for the Reidemeister 2 move $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the initial state $\psi$, we have that the solution to Schrödinger’s equation for time $t$ is:

$$e^{itH_g} \psi = \begin{pmatrix} \cos \left( \frac{\pi}{4} \right) \\ \cos \left( \frac{3\pi}{4} \right) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

**Some Miscellaneous Unitary Transformations Not in $A(n)$**

- **The crossing tunneling transformation**
  $$\tau_{ij} = \begin{pmatrix} 0 & (i,j) \\ (i,j) & 0 \end{pmatrix}$$

- **The mirror image transformation**
  $$\mu = \prod_{i,j} \begin{pmatrix} 0 & (i,j) \\ (i,j) & 0 \end{pmatrix}$$

**Observables which are Quantum Knot Invariants**
Question. What do we mean by a physically observable knot invariant?

Let \( (\mathcal{K}^{(n)}, A(n)) \) be a quantum knot system. Then a quantum observable \( \Omega \) is a Hermitian operator on the Hilbert space \( \mathcal{K}^{(n)} \).

But which observables \( \Omega \) are actually knot invariants?

Definition. An observable \( \Omega \) is an invariant of quantum knots provided \( U \Omega U^{-1} = \Omega \) for all \( U \in A(n) \).

Theorem. Let \( (\mathcal{K}^{(n)}, A(n)) \) be a quantum knot system, and let \( \Omega \) be an observable on \( \mathcal{K}^{(n)} \). Let \( St(\Omega) \) be the stabilizer subgroup for \( \Omega \), i.e.,

\[
St(\Omega) = \left\{ U \in A(n) : U \Omega U^{-1} = \Omega \right\}
\]

Then the observable

\[
\sum_{U \in St(\Omega)} U \Omega U^{-1}
\]

is a quantum knot invariant, where the above sum is over a complete set of coset representatives of \( St(\Omega) \) in \( A(n) \).

The following is an example of a quantum knot invariant observable:

\[
\Omega = \left| \begin{array}{c}
\bigotimes_{i} \bigotimes_{j} \bigotimes_{k} \bigotimes_{l}
\end{array} \right| + \left| \begin{array}{c}
\bigotimes_{i} \bigotimes_{j} \bigotimes_{k} \bigotimes_{l}
\end{array} \right|
\]

Future Directions & Open Questions
Future Directions & Open Questions

- What is the structure of the ambient group $A(n)$ and its direct limit $A = \lim A(n)$? Can one find a presentation of this group?
- Unlike classical knots, quantum knots can exhibit the non-classical behavior of quantum superposition and quantum entanglement. Are quantum and topological entanglement related to one another? If so, how?

Future Directions & Open Questions

- How does one find a quantum observable for the Jones polynomial?
- How does one create quantum knot observables that represent other knot invariants such as, for example, the Vassiliev invariants?

Future Directions & Open Questions

- What is gained by extending the definition of quantum knot observables to POVMs?
- What is gained by extending the definition of quantum knot observables to mixed ensembles?

Future Directions & Open Questions

Def. We define the mosaic number of a knot $k$ as the smallest integer $n$ for which $k$ is representable as a knot $n$-mosaic.

- The mosaic number of the trefoil is 4. In general, how does one compute the mosaic number of a knot?
- Is the mosaic number related to the crossing number of a knot?

Future Directions & Open Questions

- Can quantum knot systems be used to model and predict the behavior of
  - Quantum vortices in supercooled helium 2?
  - Quantum vortices in the Bose-Einstein Condensate
  - Fractional charge quantification that is manifest in the fractional quantum Hall effect

UMBC Quantum Knots Research Lab

This question can be answered in the positive by finding a Hamiltonian $H$ for a quantum knot that predicts the behavior of a quantum vortex.
We would like to find an equivalent definition of quantum knots that is more directly related to the spatial configurations of knots in 3-space.

**Rationale:** Mosaics are based on knot projections, and hence, only indirectly on the actual knot. Moreover, PI & Reidemeister moves are moves on knot projections, and hence, also only indirectly associated with the actual spatial configuration of knots.
Can we find an alternative approach to knot theory?

Before we ask this question, we need to find the answer to a more fundamental question:

How does a dog wag its tail?

How does a dog wag its tail?

My best friend Tazi knew the answer.

Tazi = Tasmanian Tiger

Key Intuitive Idea

A curve in 3-space has 3 local (i.e., infinitesimal) degrees of freedom.

Wiggle = A curvature move
Wag = A torsion move
Tug = A metric move

Can we take this idea and use it to create a useable well-defined set of moves which will replace the Reidemeister moves?

Zawitz's Tangle

Moves only by Wagging
How does a dog wag its tail?

Yes, when Tazi moved her tail, she naturally understood how a curve can move in 3-space!

She had a keen understanding of differential geometry.

Differential Geometry: The Frenet Frame

Each point of a curve in 3-space is naturally associated with a 3-frame, called the Frenet frame.

\[ B = \text{binormal} = T \times N \]

- A curve **bends** by rotating about \( B \) – as measured by its **curvature** \( K \).
- A curve **twists** by rotating about \( N \) – as measured by its **torsion** \( \tau \).
- A curve stretches or contracts along its tangent \( T \).
Key Intuitive Idea

A curve in 3-space has 3 local (i.e., infinitesimal) degrees of freedom.

Can we take this idea and use it to create a useable well-defined set of moves which will replace the Reidemeister moves?

Clues from Mechanical Engineering

**Linkage** = Inextensible bars (i.e., rods) connected by joints

**Joints**:  
* Planar  
* Spherical  
* Slider

Mechanisms

**4-Bar Linkage**  
All Joints Planar

Since endpoints fixed, this is a local move on linkages. The rest of the linkage is untouched

This is a local curvature move, taking place in a fixed plane. We call it a **wiggle**.

**3-Bar Linkage**  
All Joints Spherical

Since endpoints fixed, this is a local move on linkages. The rest of the linkage is untouched

This is a local torsion move, locally twisting the linkage into a new plane. We call it a **wag**.

**4-Bar Slider**  
All Joints Planar except Slider

Since endpoints fixed, this is a local move on linkages. The rest of the linkage is untouched

This is a local expansion/contraction move, taking place in a fixed plane. We call it a **tug**.
Translating M.E. into Knot Theory

**Definition:** Two piecewise linear (PL) knots $K_1$ and $K_2$ are said to be of the same knot type, written $K_1 \sim K_2$, provided one can be transformed into the other by a finite sequence of the following local moves:

1) *A tug*:

2) *A wiggle*:

3) *A wag*:

Using the methods found in Reidemeister’s proof of the completeness of the Reidemeister moves, we have:

**Theorem:** Wiggles and wags can be expressed as sequences of tugs.

In fact, Reidemeister’s fundamental move was essentially a tug.

So why bother with wiggles and wags?

Why Wiggle & Wag?

My reason is that, while investigating electromagnetic knots, the knot theoretic tools I needed to study knots that naturally arise in physics were simply not available.

What was needed was a knot theory for inextensible knots. Reidemeister’s moves, which are essentially derived from the tug move, are simply NOT inextensible moves.

Inextensible Knot Theory

**Definition:** Two piecewise linear (PL) knots $K_1$ and $K_2$ are said to be of the same inextensible knot type, written $K_1 \approx K_2$, provided one can be transformed into the other by a finite sequence of wiggles and wags.

**Proposition.** Let $K_1$ and $K_2$ be PL knots. Then

$$K_1 \approx K_2 \iff [K_1] = [K_2]$$

So it would seem that we have gained NOTHING by creating inextensible knot theory!!

But think again!

Lomonaco, Samuel J., Jr., The modern legacies of Thomson’s atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 – 166.
Inextensible Knot Theory

Because of this modified definition, we will be able to:

• Create infinitesimal knot moves
• Create knot move differential forms
• Take variational derivatives with respect to these infinitesimal moves
• And much more

Lattice Knots

The Cubic Honeycomb

A Scaffolding for 3-Space

For each non-negative integer \( \ell \), let \( L_\ell \) denote the 3-D lattice of points

\[
L_\ell = \{ \left( \frac{m_1}{2^{\ell}}, \frac{m_2}{2^{\ell}}, \frac{m_3}{2^{\ell}} \right) : m_1, m_2, m_3 \in \mathbb{Z} \}
\]

lying in Euclidean 3-space \( \mathbb{R}^3 \).

This lattice determines a tiling of \( \mathbb{R}^3 \) by

\( 2^{-\ell} \times 2^{-\ell} \times 2^{-\ell} \) cubes,

called the cubic honeycomb of \( \mathbb{R}^3 \) (of order \( \ell \)).

The Cubic Honeycomb

A Scaffolding for 3-Space

We think of this honeycomb as a cell complex \( G_\ell \) for \( \mathbb{R}^3 \) consisting of:

- Vertices
- Edges
- Faces
- Cubes

\( a \in L_\ell \)

\( F \) Open

\( F \) Open

\( B \) Open

All cells of positive dimension are open cells.

Lattice Knots

Definition. A lattice graph \( G \) (of order \( \ell \)) is a finite subset of edges (together with their endpoints) of the honeycomb \( G_\ell \).

Definition. A lattice Knot \( K \) (of order \( \ell \)) is a lattice 2-valent graph (of order \( \ell \)). Let \( \mathcal{K}^{(\ell)} \) denote the set of all lattice knots of order \( \ell \).
We define an orientation of $\mathbb{R}^3$ by selecting a right handed frame at the origin $O = (0,0,0)$ properly aligned with the edges of the honeycomb, and by parallel transporting it to each vertex $a \in L$.

We refer to this frame as the preferred frame.

A vertex $a$ of a cube $B$ is called a preferred vertex of $B$ if the first octant of the preferred frame at $a$ contains the cube $B$.

Since $B$ is uniquely determined by its preferred vertex, we use the notation $B = B^{(t)}(a)$. 

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**Lattice Knots**

Lattice Trefoil  
Lattice Hopf Link

**Necessary Infrastructure**

**Orientation of 3-Space**

**Color Coding Conventions for Vertices & Edges**

- **Solid Red**: Part of the Lattice Knot
- **“Hollow” Gray**: Not part of Lattice Knot
- **Solid Gray**: Indeterminant, maybe part of Lattice Knot
The preferred edges and preferred faces of $B^{(t)}(a)$ are respectively the edges and faces of $B^{(t)}(a)$ that have $a$ as a vertex.

- Every edge is a preferred edge of exactly one cube.
- Every face is the preferred face of exactly one cube.

Hence, the following notation uniquely identifies each edge and face of the cell complex $\mathcal{C}_a$.

\[
E_p^{(t)}(a) = \text{Preferred face parallel to } e_p
\]

\[
F_p^{(t)}(a) = \text{Preferred face perpendicular to } e_p
\]

**Drawing Conventions**

When drawn in isolation, each cube $B^{(t)}(a)$ is drawn with edges parallel to the preferred frame, and with the preferred vertex in the back bottom left hand corner.

When drawn in isolation, $B^{(t)}(a)$ is always drawn with preferred vertex in the upper left hand corner, and with $e_p(a)$ pointing out of the page.

**The Left and Right Permutations**

Define the left and right permutations $\mathbb{L}$ and $\mathbb{R}$ as:

\[
\mathbb{L}: \{1,2,3\} \mapsto \{1,2,3\}
\]

\[
1 \mapsto 2
\]

\[
2 \mapsto 3
\]

\[
3 \mapsto 1
\]

\[
\mathbb{R}: \{1,2,3\} \mapsto \{1,2,3\}
\]

\[
1 \mapsto 3
\]

\[
2 \mapsto 1
\]

\[
3 \mapsto 2
\]

Ergo,

\[
e_p = e_1 \times e_2
\]

\[
e_L = e_2 \times e_3
\]

\[
e_R = e_3 \times e_1
\]
Vertex Translation

Let $a$ be a vertex in the lattice $\mathcal{L}$

$$a^p = a + 2^t e_p$$
$$a^r = a - 2^t e_p$$
$$a^l = a + 3 \cdot 2^t e_p$$

So for example,

$$a^{2^3} = a + 2 \cdot 2^t e_i - 5 \cdot 2^t e_j + 2^t e_k$$

The Preferred Vertex (PV) Map $[-]$.

$$B^0(a) \cup \{a\} \cup \{E_p(a)\} \cup \{E_{\parallel}(a)\}$$

Half Closed Cube

The Preferred Vertex Map

$$[-] : \mathbb{R}^3 \rightarrow \mathcal{L}$$

$$x = (x_1, x_2, x_3) \mapsto \{2^{-t}[2^t x_1], 2^{-t}[2^t x_2], 2^{-t}[2^t x_3]\}$$

Lattice Knot Moves

Definition. A lattice graph $G$ (of order $\ell$) is a finite subset of edges (together with their endpoints) of the honeycomb $\mathcal{G}$.

Definition. A lattice Knot $K$ (of order $\ell$) is a lattice 2-valent graph (of order $\ell$). Let $\mathcal{K}^{(\ell)}$ denote the set of all lattice knots of order $\ell$. 

Lattice Knots

Lattice Knots

Lattice Trefoil

Lattice Hopf Link
**Lattice knot moves**

**Definition.** A lattice knot move $\mu$ (of order $\ell$) is a bijection $\mu : \mathcal{R}^{(\ell)} \rightarrow \mathcal{R}^{(\ell)}$

The move $\mu$ is said to be local if there exists a cube $B^{(\ell)}(a)$ in the lattice such that $\mu(K) - B^{(\ell)}(a) = K - B^{(\ell)}(a)$ for all $K \in \mathcal{R}^{(\ell)}$

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**Tugs**

$L_1^{(\ell)}(a, p, \square) = \square^{(\ell)}(a, p)$

This is a local move on face $F_p^{(\ell)}(a)$

For each cube, 4 Tugs for each preferred face

12 tugs for each cube

**Tugs**

$L_1^{(\ell)}(a, p, \square) = \square^{(\ell)}(a, p)$

For each cube, 2 Wiggles for each preferred face

6 wiggles per cube

**Wiggles**

Wiggles are inextensible local moves
Please recall that the left and right permutations are defined as:

\[ L: \{1,2,3\} \rightarrow \{1,2,3\}, \quad 1 \mapsto 2, \quad 2 \mapsto 3, \quad 3 \mapsto 1 \]

\[ R: \{1,2,3\} \rightarrow \{1,2,3\}, \quad 1 \mapsto 3, \quad 2 \mapsto 1, \quad 3 \mapsto 2 \]

\[ e_p = e'_L \times e'_{\parallel} \]

\[ e_L = e_{\parallel} \times e_p \]

\[ e_{\parallel} = e_p \times e'_L \]

For each cube, there are 4 wags for each of its preferred faces. Hence, there are 12 wags per cube.

Wags are inextensible local moves.
An Example of a Wiggle Move

$K(a)$

The Ambient Groups

The Ambient Groups $\Lambda_{\ell}$ and $\tilde{\Lambda}_{\ell}$

Definition. The ambient group $\Lambda_{\ell}$ is the group generated by tugs, wiggles, and wags of order $\ell$.

Definition. The inextensible ambient group $\tilde{\Lambda}_{\ell}$ is the group generated only by wiggles and wags of order $\ell$.

Tugs, wiggles, and wags are a set of involutions that generate the above groups.

Tug, Wiggle, & Wag are Permutations

For each $\ell \geq 0$, each of the above moves, Tug, Wiggle, & Wag, is a permutation (bijection) on the set $\mathcal{K}(\ell)$ of all lattice knots of order $\ell$.

In fact, each of the above local moves, as a permutation, is the product of disjoint transpositions.

What Is the Ambient Group?

What is the ambient group???

Observation: Wiggle, wag, and tug are symbolic conditional moves, as are the Reidemeister moves.

For example, the tug

$$\mathcal{U}(a,p) = \begin{cases} (K - \text{Conditions}) \cup \text{otherwise} & \text{if } K \cap \text{Conditions} = \text{otherwise} \\ (K - \text{Conditions}) \cup \text{otherwise} & \text{otherwise} \end{cases}$$
Observation: Each is a symbolic representation of an authentic conditional move, i.e., a conditional orientation preserving (OP) auto-homeomorphism of \( \mathbb{R}^3 \).

Moreover, each involved (OP) auto-homeomorphism \( h: \mathbb{R}^3 \to \mathbb{R}^3 \) is local, i.e., there exists a 3-ball \( D \) such that
\[
\delta|_{\mathbb{R}^3 \setminus D} = \text{id} : \mathbb{R}^3 \setminus D \to \mathbb{R}^3 \setminus D.
\]

Def. A local authentic conditional (LAC) move on a family of knots \( \mathcal{F} \) is a map
\[
\Phi: \mathcal{F} \to \text{LAC}_\text{op}(\mathbb{R}^3)
\]

such that
\[
\Phi_k(K) \in \mathcal{F}, \quad \forall K \in \mathcal{F}.
\]

Let \( \text{LAC}_\text{op}(\mathbb{R}^3) \) be the space of all LAC moves for the family \( \mathcal{F} \).

Proposition. \( \text{LAC}_\text{op}(\mathbb{R}^3) \) is a monoid.

In the paper "Quantum Knots and Lattices," we construct a faithful representation
\[
\Gamma: \Lambda \to \text{LAC}_\text{op}(\mathbb{R}^3)
\]

into a subgroup of the monoid \( \text{LAC}_\text{op}(\mathbb{R}^3) \) by mapping each generator wiggle, wag, and tug onto a local conditional OP auto-homeomorphism of \( \mathbb{R}^3 \).

Refinement
The refinement injection

**Def.** We define the refinement injection $Q: \mathcal{K}^{(\ell)} \rightarrow \mathcal{K}^{(\ell+1)}$ from lattice knots of order $\ell$ to lattice knots of order $\ell+1$ as

$$Q: \mathcal{K}^{(\ell)} \rightarrow \mathcal{K}^{(\ell+1)}$$

$$K \rightarrow \bigcup_{a \in \Lambda} \bigcup_{a' \in \Lambda} \left\{ E^{(\ell)}(a), E^{(\ell)}(a') \right\}$$

**An example:**

The Refinement Morphism $Q: \Lambda_\ell \rightarrow \Lambda_{\ell+1}$

We conjecture the existence of a refinement monomorphism $Q: \Lambda_\ell \rightarrow \Lambda_{\ell+1}$ which preserves the action

$$\Lambda_\ell \times \mathcal{K}^{(\ell)} \rightarrow \mathcal{K}^{(\ell)}$$

$$(g, K) \mapsto gK$$

i.e., with the property

$$Q(g)Q(K) = Q(gK)$$

In fact, we have a construction which we believe is such a monomorphism.

Knot Type
Lattice Knot Type

Two lattice knots \( K_1 \) and \( K_2 \) in \( \mathbb{R}^n \) are of the same \( \ell \)-type, written \( K_1 \sim K_2 \), provided there is an element \( g \in \Lambda_\ell \) such that \( gK_1 = K_2 \).

They are of the same knot type, written \( K_1 \sim K_2 \), provided there is a non-negative integer \( m \) such that \( Q^m K_1 \sim Q^m K_2 \).

Inextensible Lattice Knot Type

Two lattice knots \( K_1 \) and \( K_2 \) in \( \mathbb{R}^n \) are of the same inextensible \( \ell \)-type, written \( K_1 \approx K_2 \), provided there is an element \( g \in \tilde{\Lambda}_\ell \) such that \( gK_1 = K_2 \).

They are of the same inextensible knot type, written \( K_1 \approx K_2 \), provided there is a non-negative integer \( m \) such that \( Q^m K_1 \approx Q^m K_2 \).

n-Bounded Lattices, Lattice Knots, and Ambient Groups

In preparation for creating a definition of physically implementable quantum knot systems, we need to work with finite mathematical objects.

n-Bounded Lattices

Let \( \ell \) and \( n \) be be non-negative integers. We define the \( n \)-bounded lattice of order \( \ell \) as \( \mathcal{L}_{\ell,n} = \{ a \in \mathcal{L}_\ell : |a| \leq n \} \) where \( |a| = \max(|a_i|) \).

We also have:
- \( \mathcal{E}_{\ell,n} \): The corresponding cell complex
- \( \mathcal{E}_{\ell,n}^j \): The corresponding \( j \)-skeleton

n-Bounded Lattice Knots & Ambient Groups

\( \mathbb{R}^{(\ell,n)} = \mathbb{R}^n \cap \mathcal{L}_{\ell,n} \) set of \( n \)-bounded lattice knots of order \( \ell \)

\( \Lambda_{\ell,n} = \Lambda_\ell \big|_{\mathcal{L}_{\ell,n}} \) Ambient group of order \( (\ell,n) \)

\( \tilde{\Lambda}_{\ell,n} = \tilde{\Lambda}_\ell \big|_{\mathcal{L}_{\ell,n}} \) Inextensible Ambient group of order \( (\ell,n) \)
We also have the injection
\[ i: \mathfrak{L}(e^L, e^{L+1}) \to \mathfrak{L}(e^L, e^{L+1}) \]
and the monomorphisms
\[ i \circ \Lambda(e^L) \to \Lambda(e^{L+1}) \quad \text{and} \quad i \circ \tilde{\Lambda}(e^L) \to \tilde{\Lambda}(e^{L+1}) \]

We thus have a nested sequence of lattice knot systems
\[ (\mathfrak{L}(e^L, \Lambda_{e^L})) \to (\mathfrak{L}(e^L, \Lambda_{e^L+1})) \to \cdots \to (\mathfrak{L}(e^L, \Lambda_{e^n})) \to \cdots \]

Two lattice knots \( K_1 \) and \( K_2 \) in \( \mathfrak{L}(e^L, e^{L+1}) \) are said to be of the same lattice knot \((L, n)\)-type, written \( K_1 \sim K_2 \)
provided there is an element \( g \in \Lambda_{e^L} \) such that \( gK_1 = K_2 \)

They are of the same lattice knot type, written \( K_1 \approx K_2 \)
provided there are non-negative integers \( L' \) and \( n' \) such that \( L' + n' \)

Quantum Knots

It’s time to remodel the bounded lattice \( \mathcal{L}_{e,n} \) by painting all its edges.

Two available cans of paint
- “Solid” Red
- “Hollow” Gray

Set of all 2-colorings of edges of \( \mathcal{L}_{e,n} \) Identification

Edge Coloring Space \( \mathcal{E} \)

\[ \mathcal{E} = \text{2-D Hilbert space with orthonormal basis} \]

\[ \langle 0 \rangle = \text{Non-Edge} \]

\[ \langle 1 \rangle = \text{Existant Edge} \]

Quantum Hilbert Space of Lattice Graphs in \( \mathcal{L}_{e,n} \)

\[ \mathcal{H}(e^n) = \bigotimes_{e^L \mathcal{E} \mathcal{G}(e^n)}(\mathcal{E}) \]
Quantum Knots

Hilbert Space of Lattice Graphs in $\mathcal{L}_{n}$

$\mathcal{G}^{(n)} = \bigotimes_{E \in \text{E} \cap \mathcal{L}_{n}} E$

Orthonormal basis is:

$\left\{ |G\rangle \mid G \text{ lattice graph in } \mathcal{L}_{n} \right\}$

which is identified with

$\left\{ |G\rangle \mid G \text{ lattice graph in } \mathcal{L}_{n} \right\}$

Quantum Knots

Hilbert Space of quantum knots

$\mathcal{K}^{(n)} = \text{Sub-Hilbert space of } \mathcal{G}^{(n)} \text{ with orthonormal basis}$

$\left\{ |K\rangle \mid K \in \mathcal{K}^{(n)} \right\}$

An Example of a Quantum Knot

$|K\rangle = \sqrt{2}$

The Ambient Group $\Lambda_{n}$ as a Unitary Group

We identify each element $g \in \Lambda_{n}$ with the linear transformation defined by

$\mathcal{K}^{(n)} \rightarrow \mathcal{K}^{(n)}$

$|K\rangle \mapsto |gK\rangle$

Since each element $g \in \Lambda_{n}$ is a permutation, it is a linear transformation that simply permutes basis elements.

Hence, under this identification, the ambient group $\Lambda_{n}$ becomes a discrete group of unitary transforms on the Hilbert space $\mathcal{K}^{(n)}$.

An Example of the $\Lambda_{n}$ Group Action

$\gamma^{-1}$

$\text{Vertex } a^{-1}$

$\text{Face } E_{n}(w^i)$

$\text{Wiggle } \Box^{a}(w^i, x)$

$\Box^{a}(w^i, x)(|K\rangle) = \sqrt{2}$
**The Quantum Knot System \((\mathcal{K}^{(\ell,n)}, \Lambda_{\ell,n})\)**

**Def.** A quantum knot system \((\mathcal{K}^{(\ell,n)}, \Lambda_{\ell,n})\) is a quantum system having \(\mathcal{K}^{(\ell,n)}\) as its state space, and having the Ambient group \(\Lambda_{\ell,n}\) as its set of accessible unitary transformations.

The states of quantum system \((\mathcal{K}^{(\ell,n)}, \Lambda_{\ell,n})\) are quantum knots. The elements of the ambient group \(\Lambda(n)\) are quantum moves.

\[
\mathcal{K}^{(\ell)} \xrightarrow{\ell} \cdots \xrightarrow{\ell} \mathcal{K}^{(\ell,n)} \xrightarrow{\ell} \mathcal{K}^{(\ell+1,n)} \xrightarrow{\ell} \cdots
\]

**Physically Implementable**

Choosing integers \(\ell\) and \(n\) is analogous to choosing respectively the thickness and the length of the rope. The smaller the thickness and the longer the rope, the more knots that can be tied.

The parameters (wiggle, wag, & tug) one turns to spatially manipulate the quantum knot.

**Physics**

Implementable

Choosing integers \(\ell\) and \(n\) is analogous to choosing respectively the thickness and the length of the rope. The smaller the thickness and the longer the rope, the more knots that can be tied.

The parameters (wiggle, wag, & tug) one turns to spatially manipulate the quantum knot.

**Def.** Two quantum knots \(K_1\) and \(K_2\) are of the same knot --type, written

\[
K_1 \sim \ell \leftrightarrow K_2,
\]

provided there is an element \(g \in \Lambda_{\ell,n}\) s.t.

\[
gK_1 = K_2.
\]

They are of the same knot type, written

\[
K_1 = K_2,
\]

provided there are integers \(\ell', n' \geq 0\) such that

\[
\mathcal{Q}'^{\ell'}|K_1\rangle \sim_{\ell'} \mathcal{Q}'^{n'}|K_2\rangle
\]

**Two Quantum Knots of the Same Knot Type**

**Wiggle**

\[
\mathcal{Q}'^\ell(a',3): |K\rangle = \frac{1}{\sqrt{2}} \left( |\square\rangle + |\square'}\rangle \right)
\]

**Two Quantum Knots NOT of the Same Knot Type**

\[
K_1 = \frac{1}{\sqrt{2}} \left( |\square\rangle + |\square\rangle \right)
\]

\[
K_2 = |\square\rangle + |\square\rangle
\]

**Hamiltonians of the Generators of the Ambient Group**
Hamiltonians for $A(n)$

Each generator $g \in \Lambda_{t,3}$ is the product of disjoint transpositions, i.e.,

$$g = (K_{a_1} K_{b_1})(K_{a_1} K_{b_2}) \cdots (K_{a_1} K_{b_r})$$

Choose a permutation $\eta$ so that

$$\eta^{-1}g \eta = (K_{a_1} K_{b_1})(K_{a_1} K_{b_2}) \cdots (K_{a_1} K_{b_r})$$

Hence,

$$\eta^{-1} = \begin{pmatrix} \sigma_1 & \cdots & 0 \\ 0 & \cdots & \sigma_r \end{pmatrix}$$

where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The Log of a Unitary Matrix

Let $U$ be an arbitrary finite $r \times r$ unitary matrix.

Then eigenvalues of $U$ all lie on the unit circle in the complex plane.

Moreover, there exists a unitary matrix $W$ which diagonalizes $U$, i.e., there exists a unitary matrix $W$ such that

$$WUW^{-1} = \Delta(e^{i \theta_1}, e^{i \theta_2}, \ldots, e^{i \theta_r})$$

where $e^{i \theta_1}, e^{i \theta_2}, \ldots, e^{i \theta_r}$ are the eigenvalues of $U$.

Hamiltonians for $A(n)$

Also, let $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and note that

$$\ln(\sigma) = \frac{i \pi}{2} (2s + 1)(\sigma_s - \sigma_r), \quad s \in \mathbb{Z}$$

For simplicity, we always choose the branch $s = 0$.

$$H_s = -i \eta \ln(\eta^{-1}g \eta) \eta^{-1}$$

The Log of a Unitary Matrix

Then

$$\ln(U) = W^{-1} \Delta \left( \ln(e^{i \theta_1}), \ln(e^{i \theta_2}), \ldots, \ln(e^{i \theta_r}) \right) W$$

Since $\ln(e^{i \theta}) = i \theta + 2\pi in_j$, where $n_j \in \mathbb{Z}$ is an arbitrary integer, we have

$$\ln(U) = iW^{-1} \Delta(\theta_1 + 2\pi n_1, \theta_2 + 2\pi n_2, \ldots, \theta_r + 2\pi n_r)W$$

where $n_1, n_2, \ldots, n_r \in \mathbb{Z}$

The Log of a Unitary Matrix

Since $e^{i \theta} = \sum_{n=0}^{\infty} A^n / (m!)$, we have

$$e^{i \eta} = e^{-i \Delta(e^{i \theta_1}, \ldots, e^{i \theta_r})}$$

$$= W^{-1} e^{i \Delta(e^{i \theta_1}, \ldots, e^{i \theta_r})} W$$

$$= W^{-1} \Delta(e^{i \theta_1}, \ldots, e^{i \theta_r}) W$$

$$= W^{-1} \Delta(e^{i \theta_1 + 2\pi n_1}, \ldots, e^{i \theta_r + 2\pi n_r}) W$$

$$= W^{-1} \Delta(e^{i \theta_1}, \ldots, e^{i \theta_r}) W = U$$
**Observables which are Quantum Knot Invariants**

**Question.** What do we mean by a physically observable knot invariant?

Let \((K^{(n)}, \Lambda_{t,n})\) be a quantum knot system. Then a quantum observable \(\Omega\) is a Hermitian operator on the Hilbert space \(K^{(n)}\).

**Definition.** An observable \(\Omega\) is an invariant of quantum knots provided \(U \Omega U^{-1} = \Omega\) for all \(U \in \Lambda_{t,n}\).

**Theorem.** Let \((K^{(n)}, \Lambda_{t,n})\) be a quantum knot system, and let \(\Lambda_{t,n} = \bigoplus W_r\) be a decomposition of the representation \(\Lambda_{t,n} \times K^{(n)} \to K^{(n)}\) into irreducible representations.

Then, for each \(r\), the projection operator \(P_r\) for the subspace \(W_r\) is a quantum knot observable.

**Question.** But how do we find quantum knot invariant observables?

In \(K^{(n)}\), the following is an example of an inextensible quantum knot invariant observable:

\[
\Omega = \sum_{\rho = 1}^r \langle \delta F^{(n)}_\rho (a) | \delta F^{(n)}_\rho (a) \rangle + \sum_{\rho = 1}^r \sum_{a'' = 1}^{a''} \langle \delta F^{(n)}_\rho (a'') | \delta F^{(n)}_\rho (a'') \rangle
\]

where \(\delta F^{(n)}_\rho (a)\) denotes the boundary of the face \(F^{(n)}_\rho (a)\).
Future Directions & Open Questions

• What is the structure of the ambient groups $\Lambda_1, \Lambda_2, \Lambda_n, \Lambda_{\infty}$, and their direct limits? Can one find a presentation of these groups? Are they Coxeter groups?

• Exactly how are the lattice and the mosaic ambient groups related to one another?

Future Directions & Open Questions

• Unlike classical knots, quantum knots can exhibit the non-classical behavior of quantum superposition and quantum entanglement. Are quantum and topological entanglement related to one another? If so, how?

Future Directions & Open Questions

• How does one find a quantum observable for the Jones polynomial? How does one create quantum knot observables that represent other knot invariants such as, for example, the Vassiliev invariants?

This would be a family of observables parameterized by points on the circle in the complex plane. Does this approach lead to an algorithmic improvement to the quantum algorithm created by Aharonov, Jones, and Landau?

Future Directions & Open Questions

• What is gained by extending the definition of quantum knot observables to POVMs?

• What is gained by extending the definition of quantum knot observables to mixed ensembles?

Future Directions & Open Questions

Def. We define the lattice number of a knot $K$ as the smallest integer $n$ for which $K$ is representable as a lattice knot of order $(\ell=0,n)$

How does one compute the lattice number of a knot? How does one find a quantum observable for the mosaic number?

Lattice number related to the mosaic number of a knot?
Future Directions & Open Questions

Quantum Knot Tomography: Given many copies of the same quantum knot, find the most efficient set of measurements that will determine the quantum knot to a chosen tolerance $\varepsilon > 0$.

Quantum Braids: Use lattices to define quantum braids. How are such quantum braids related to the work of Freedman, Kitaev, et al on anyons and topological quantum computing?

Tug, Wiggle, & Wag are “Physics Friendly”

Variational Derivatives w.r.t. moves

The Preferred Vertex (PV) Map $[-]$

Since the half closed cubes form a partition of 3-space $\mathbb{R}^3$, we have the preferred vertex map $PV^{(t)}: \mathbb{R}^3 \rightarrow A_2 = \{\left(\frac{m_1}{2^l}, \frac{m_2}{2^l}, \frac{m_3}{2^l}\right) : m_1, m_2, m_3 \in \mathbb{Z}\}$.
A finitely piecewise smooth (FPS) knot is a knot $x$ in 3-space $\mathbb{R}^3$ that consists of finitely many piecewise smooth ($C^\infty$) segments with no two consecutive segments meeting in a tangential cusp.

Let $\mathcal{F}_{FPS}$ be the family of FPS knots.

Let $F : \mathcal{F}_{FPS} \to \mathbb{R}$ be a real valued functional on $\mathcal{F}_{FPS}$.

Let $\mathcal{F}_{FPS} = \{ x \}$ be a real valued functional on $\mathcal{F}_{FPS}$.

Total Tug, Total Wiggle, & Total Wag

Total Tug:
$$\lim_{\delta \to 0} \frac{F(\square^{(t)}(x), f) - F(x)}{2^{-\delta}}$$

Total Wiggle:
$$\lim_{\delta \to 0} \frac{F(\square^{(t)}(x), f) - F(x)}{2^{-\delta}}$$

Total Wag:
$$\lim_{\delta \to 0} \frac{F(\square^{(t)}(x), f) - F(x)}{2^{-\delta}}$$

Variational Derivatives w.r.t. moves

Conjecture: A functional $F : \mathcal{F}_{FPS} \to \mathbb{R}$ is a knot invariant if all its variational derivatives exist and are zero.

Quantum Knots Research Lab

We at UMBC are very proud of our new state of the art Quantum Knots Research Laboratory. We have just purchased some of the latest and most advanced equipment in quantum knots research !!!