A QUICK ENTRY INTO QUANTUM COMPUTATION
*** DRAFT ***

SAMUEL J. LOMONACO

ABSTRACT. The purpose of these class notes is to give students a quick entry into the field of quantum computing so that they will be able to focus more of their time on more advanced aspects of quantum computation and quantum information science.

CONTENTS

1. Introduction

To be completed.

Minimum background assumed throughout this handout: *Undergraduate courses in linear algebra and in algorithms.* Reasonable proficiency with the field of complex numbers $\mathbb{C}$.

2. The Classical Shannon Bit

Definition 1. *A **Shannon bit**, simply called a **bit**, is a classical physical system* $\mathbb{S}$ *whose state* $\psi = \psi(\mathbb{S})$ *is either* 0 *or* 1*. The set* $\{0, 1\}$ *is called the **state space** of the bit* $\mathbb{S}$*.*

Example 1. *A two sided coin is an example of Shannon bit, with "heads" and "tails" respectively representing "0", and "1". Another example would be a flipflop on a silicon waffer. Can you think of other examples?*

Date: September 5, 2021.

Definition 2. *Let n be a positive integer. A* ***multipartite bit system*** *$\mathbb{S}$ is a collection*

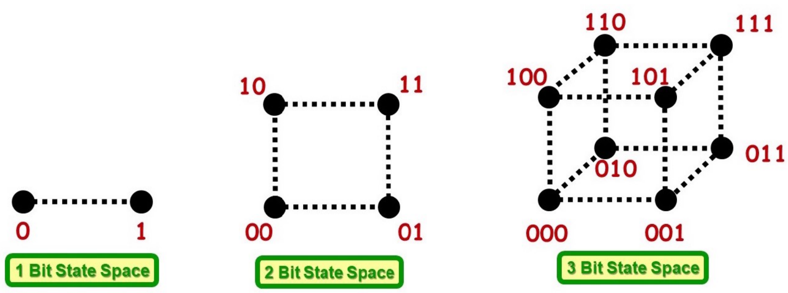
$$\mathbb{S} = \{\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_n\}$$

of n bits. The ***state*** *$\psi = \psi(\mathbb{S})$ of $\mathbb{S}$ is an n bit string $\psi \in \{0,1\}^n$. The collection*

$$\{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \ldots \times \{0,1\}}_{n}$$

of n bit strings is called the ***state space*** *of the multipartite bit system $\mathbb{S}$.*

Remark 1. *Please note that the the number of states of an n bit multipartite system is $O(2^n)$. But each state can be represented by an $O(n)$ bit string.*



3. The Qubit

Definition 3. *A* ***quantum bit****, simply called a* ***qubit****, is a quantum system $\mathcal{Q}$ whose state $\psi = \psi(\mathcal{Q})$ is a vector in a Hilbert space $\mathcal{H}$. The Hilbert space $\mathcal{H}$ is called the* ***state space*** *of $\mathcal{Q}$.*

Question: But what is a Hilbert space?

Before we can proceed further, we need to discuss **Hilbert spaces** and **Dirac notation**, a.k.a., **Bra-Ket notation**.

4. Hilbert spaces, and Dirac notation

Remark 2. *When first reading this handout, a student may find it easier to read the definition of Hilbert space given below, and then skip to the next section. Then after becoming more comfortable with the subject, he/she should return and read this entire section.*

As stated earlier, the state space of a quantum system is a Hilbert space, which is now defined as follows:

Definition 4. *A **Hilbert space** $\mathcal{H}$ is a vector space over the field of complex numbers $\mathbb{C}$ together with an inner product*

$$\begin{array}{rcc} \cdot : \mathcal{H} \times \mathcal{H} & \longrightarrow & \mathbb{C} \\ (\psi_1, \psi_2) & \longmapsto & \psi_1 \cdot \psi_2 \in \mathbb{C} \end{array}$$

such that, for all $\psi_1, \psi_2, \psi_3 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, the following axioms hold:

1) $(\psi_1 + \psi_2) \cdot \psi_3 = \psi_1 \cdot \psi_3 + \psi_2 \cdot \psi_3$ *and* $\psi_3 \cdot (\psi_1 + \psi_2) = \psi_3 \cdot \psi_1 + \psi_3 \cdot \psi_2$.
2) $\psi_1 \cdot (\lambda \psi_2) = \lambda (\psi_1 \cdot \psi_2)$.
3) $(\psi_1 \cdot \psi_2)^* = \psi_2 \cdot \psi_1$ *for $\psi_1, \psi_2 \in \mathcal{H}$, where the superscipt "$*$" denotes complex conjugation. Hence,* $(\lambda \psi_1) \cdot \psi_2 = \lambda^* (\psi_1 \cdot \psi_2)$.
4) *$\mathcal{H}$ is **topologically complete**, i.e., Cauchy sequences of vectors in $\mathcal{H}$ converge to vectors in $\mathcal{H}$.*

Remark 3. *Throughout these lecture notes, all Hilbert spaces will be assumed to be finite dimensional. Since all finite dimensional vector spaces over $\mathbb{C}$ automatically satisfy axiom 4), we can and do henceforth ignore axiom 4).*

In quantum physics, Dirac notation is most commonly used. Vectors in the state space $\mathcal{H}$ are called **kets**, and denoted by

$$z = |label\rangle \ ,$$

where "label" is again simply any label chosen to designate the ket. In addition, Dirac notation makes use of an additional Hilbert space $\mathcal{H}^\dagger$, which is the Hilbert space of all linear transformation from $\mathcal{H}$ to the complex numbers $\mathbb{C}$, i.e.,

$$\mathcal{H}^\dagger = Hom_{\mathbb{C}}(\mathcal{H}, \mathbb{C}) = \{ f : \mathcal{H} \longrightarrow \mathbb{C} \mid f \text{ linear} \}$$

with vector addition and scalar multiplication defined respectively by

$$(f_1 + f_2)(|\psi\rangle) = f_1(|\psi\rangle) + f_2(|\psi\rangle) \qquad \text{and} \qquad (\lambda f)(|\psi\rangle) = \lambda (f(|\psi\rangle)) \, ,$$

and with inner product defined[1] by

$$f_1 \cdot f_2 = \sum_{j=0}^{n-1} f_1(|\psi_j\rangle)^* f_2(|\psi_j\rangle) \ ,$$

where $|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{n-1}\rangle$ is an orthonormal basis of the state space $\mathcal{H}$.

The vectors of $\mathcal{H}^\dagger$ are called **bras** and denoted by

$$z = \langle |label\rangle| \ ,$$

where "label" is simply any label chosen to designate the bra.

There is an **adjoint map** $\mathcal{H} \xrightarrow{\dagger} \mathcal{H}^\dagger$ which maps each ket $|\psi\rangle$ into a bra designated by the same label, i.e., $|\psi\rangle^\dagger = \langle\psi|$. The bra is simply defined as the linear transformation

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\langle\psi|} & \mathbb{C} \\ |\varphi\rangle & \longmapsto & (|\psi\rangle) \cdot (|\varphi\rangle) \end{array} .$$

In other words,

$$\langle\psi| (|\varphi\rangle) = (|\psi\rangle) \cdot (|\varphi\rangle) \ ,$$

[1] The inner product $f_1 \cdot f_2$ is independent the the chosen orthonormal basis.

which is just the inner product of $\mathcal{H}$. Since $\langle\psi|\,(|\varphi\rangle)$ is such a clumsy notation, we abbreviate it as

$$\langle\psi|\varphi\rangle\ ,$$

and call it the Dirac brac-(c)-ket, or more simply, the **bracket**.

By the Riesz representation theorem, $\mathcal{H}\xrightarrow{\dagger}\mathcal{H}^{\dagger}$ is a Hilbert space anti isomorphism. Hence, every linear transformation in $\mathcal{H}^{\dagger}$ is a bra. We call $\mathcal{H}^{\dagger}$ the **dual space** of the Hilbert space $\mathcal{H}$.

Finally, since the dual space $\mathcal{H}^{\dagger}$ is a Hilbert space, we can also construct the dual $\mathcal{H}^{\dagger\dagger}$of the dual Hilbert space $\mathcal{H}^{\dagger}$. As it turns out, it can be shown that $\mathcal{H}^{\dagger\dagger}$is isomorphic to the original Hilbert space $\mathcal{H}$. So we can identify these two spaces and write

$$\mathcal{H}^{\dagger\dagger}=\mathcal{H}.$$

In this way, the adjoint operator can be seen to be an involution.

5. A more concrete description of Dirac notation

In this section, we give a description of Dirac notation that is more concrete, but by no means as complete as that given in the previous section.

We begin by considering a quantum system $\mathcal{Q}$ whose state space is a finite dimensional Hilbert space $\mathcal{H}$ of dimension n . Since all n dimensional Hilbert spaces are isomorphic to each other, we can identify $\mathcal{H}$ with the Hilbert space of n dimensional column vectors

$$z=\begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}$$

over the complex numbers $\mathbb{C}$, where vector addition is simply matrix addition

$$z+w=\begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}+\begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix}=\begin{pmatrix} z_0+w_0 \\ z_1+w_1 \\ \vdots \\ z_{n-1}+w_{n-1} \end{pmatrix}\ ,$$

where scalar multiplication is simply multiplication of a matrix by a scalar

$$\lambda z=\lambda\begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}=\begin{pmatrix} \lambda z_0 \\ \lambda z_1 \\ \vdots \\ \lambda z_{n-1} \end{pmatrix}\ ,$$

and where inner product is defined in matrix form by the matrix product

$$z \cdot w = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}^\dagger \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} z_0^* & z_1^* & \cdots & z_{n-1}^* \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix} = \sum_{j=0}^{n-1} z_j^* w_j \ ,$$

where superscript "$*$" denotes complex conjugation, and where the superscript "$\dagger$" (called the **adjoint**) denotes the **conjugate transpose** operation. In other words,

$$\begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}^\dagger = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}^{*T} = \begin{pmatrix} z_0^* \\ z_1^* \\ \vdots \\ z_{n-1}^* \end{pmatrix}^T = \begin{pmatrix} z_0^* & z_1^* & \cdots & z_{n-1}^* \end{pmatrix}$$

The column vectors of $\mathcal{H}$ will be called **kets**, and will be denoted in Dirac notation by

$$z = |z\rangle = |label\rangle \ ,$$

where "label" is simpy any conveniently chosen label used to designate the ket.

The **dual Hilbert space** $\mathcal{H}^\dagger$ of $\mathcal{H}$ is simply the Hilbert space of all row vectors

$$z = \begin{pmatrix} z_0 & z_1 & \cdots & z_{n-1} \end{pmatrix}$$

of length n over the complex numbers $\mathbb{C}$, with vector addition and scalar multiplication defined in the obvious way as matrix operations, and with inner product defined by

$$z \cdot w = \begin{pmatrix} z_0 & z_1 & \cdots & z_{n-1} \end{pmatrix} \cdot \begin{pmatrix} w_0 & w_1 & \cdots & w_{n-1} \end{pmatrix} = \begin{pmatrix} z_0 & z_1 & \cdots & z_{n-1} \end{pmatrix} \begin{pmatrix} w_0 & w_1 & \cdots & w_{n-1} \end{pmatrix}^\dagger$$

$$= \begin{pmatrix} z_0 & z_1 & \cdots & z_{n-1} \end{pmatrix} \begin{pmatrix} w_0^* & w_1^* & \cdots & w_{n-1}^* \end{pmatrix}^T = \begin{pmatrix} z_0 & z_1 & \cdots & z_{n-1} \end{pmatrix}^\dagger \begin{pmatrix} w_0^* \\ w_1^* \\ \vdots \\ w_{n-1}^* \end{pmatrix} = \sum_{j=0}^{n-1} z_j w_j^* \ .$$

The row vectors of $\mathcal{H}^\dagger$ will be called **bras**, and will be denoted in Dirac notation by

$$z = \langle z| = \langle label| \ ,$$

where "label" is simpy any conveniently chosen label used to designate the ket.

At this point, it is important to note that there is a **dual correspondance** (i.e., a **Hilbert space anti isomorphism**) between the ket space $\mathcal{H}$ and the bra space $\mathcal{H}^\dagger$ given by the **adjoint operator** $\dagger$ (a.k.a., the **conjugate transpose**) defined by

$$\begin{array}{ccc} \mathcal{H} & \overset{\dagger}{\longleftrightarrow} & \mathcal{H}^{\dagger} \\ \left|\psi\right\rangle & \longleftrightarrow & \left|\psi\right\rangle^{\dagger} = \left\langle\psi\right| \\ \left(\begin{array}{c} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{array}\right) & \longleftrightarrow & \left(\begin{array}{c} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{array}\right)^{\dagger} = \left(\begin{array}{cccc} z_0^* & z_1^* & \cdots & z_{n-1}^* \end{array}\right) \end{array}$$

With the above defined adjoint operator $\dagger$, we now see that the inner product in $\mathcal{H}$ and $\mathcal{H}^{\dagger}$ can be expressed simply as a matrix product

$$\left|\psi_1\right\rangle \cdot \left|\psi_2\right\rangle = \left(\left\langle\psi_1\right|\right)\left(\left|\psi_2\right\rangle\right) = \left\langle\psi_1\right| \cdot \left\langle\psi_2\right| \ .$$

We denote the matrix product of a bra and a ket $\left(\left\langle\psi_1\right|\right)\left(\left|\psi_2\right\rangle\right)$ by

$$\left\langle\psi_1 \middle| \psi_2\right\rangle \ ,$$

and call it the Dirac bra-(c)-ket, or more simply, the **bracket**.

Examples

To be completed

6. What is a qubit?

Definition 5. *A **quantum bit**, simply called a **qubit**, is a quantum system $\mathcal{Q}$ whose state space is a two dimensional Hilbert space $\mathcal{H}$.*

Let $\left|0\right\rangle$, $\left|1\right\rangle$ denote an arbitrarily chosen orthonormal basis of the qubit Hilbert space $\mathcal{H}$. Thus every state ket is of the form

$$\left|\psi\right\rangle = z_0\left|0\right\rangle + z_1\left|1\right\rangle \ ,$$

where z_0 and z_1 are complex numbers in $\mathbb{C}$.

In this basis,

$$\left|\psi\right\rangle = \left(\begin{array}{c} z_0 \\ z_1 \end{array}\right) \ ,$$

Thus,

$$\left|0\right\rangle = 1\left|0\right\rangle + 0\left|1\right\rangle = \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \quad \text{and} \quad \left|1\right\rangle = 0\left|0\right\rangle + 1\left|1\right\rangle = \left(\begin{array}{c} 0 \\ 1 \end{array}\right)$$

To be completed.

7. The Bloch sphere

To be completed.

8. Multipartite Quantum Systems

To be completed

References

[1] Lomonaco, Samuel J., Jr., **"A Rosetta stone for quantum mechanics with an introduction to quantum computation,"** PSAPM, vol. 58, AMS, Providence, RI, (2002), 3 - 65. [https://arxiv.org/abs/quant-ph/0007045] [PowerPoint Talk: http://www.csee.umbc.edu/~lomonaco/slides/GWU-2009-RosettaStone.pdf]

[2] Lomonaco, Samuel J., Jr., **How to Build a Device that Cannot Be Built**, Journal of Quantum Information Processing, Vol. 15, No. 3, (2016), pp. 1043-1056. DOI 10.1007/s11128-015-1206-7

University of Maryland Baltimore County (UMBC), Baltimore, MD 21250
E-mail address: `Lomonaco@umbc.edu`
URL: `http://www.csee.umbc.edu/~lomonaco`