1) Compute the addition and multiplication tables for the ring

\[ R_3 = GF(2)[x]/\langle x^3 + 1 \rangle \]

Also express each of the following ideals in the ring \( R_3 \) as a set of elements of \( R_3 \).

\( \langle 0 \rangle, \langle 1 + x \rangle, \langle x^2 + x + 1 \rangle, \langle 1 \rangle, \langle x^2 + 1 \rangle, \langle x^3 + 1 \rangle, \langle x^5 + x + 1 \rangle, \langle x^6 + 1, x^2 + x + 1 \rangle. \)

For example,

\( \langle 0 \rangle = \{0\} \) and \( \langle x^4 + x^2 + 1 \rangle = \{0, x^2 + x + 1\} \)

2) Let \( \xi \) be the primitive element of \( GF(2^6) \) defined by \( \xi = x \mod x^6 + x + 1 \). Compute the orders of the elements of \( \xi^i \) for \( i = 0, 1, ..., 62 \). Summarize your results in a log/order table. For which \( i \)'s are the \( \xi^i \)'s primitive? Do you see a pattern? Make a conjecture about this pattern. [Recall that the order of an element \( a \) is the smallest positive integer \( k \) such that \( a^k = 1 \). Also recall that an element of \( GF(2^6) \) is primitive if its order is \( 2^6 - 1 = 15 \).]

3) Let \( R \) be a commutative ring for which there exist non-zero elements \( a \) and \( b \) such that

\[ ab = 0 \]

Prove that \( R \) is not a field.

4) In the ring \( GF(2)[x] \), compute

\[ \gcd \left( x^8 + x^6 + x^4 + x^3 + x + 1, x^6 + x^5 + x^4 + x^2 + x + 1 \right) \]