Quantum Computing

Samuel J. Lomonaco, Jr.
Dept. of Comp. Sci. & Electrical Engineering
University of Maryland Baltimore County
Baltimore, MD 21250
Email: Lomonaco@UMBC.EDU
WebPage: http://www.csee.umbc.edu/~lomonaco

Overview
Four Talks

- A Rosetta Stone for Quantum Computation
- Three Quantum Algorithms
- Quantum Hidden Subgroup Algorithms
- An Entangled Tale of Quantum Entanglement

Elementary
Advanced

Lecture 4
An Entangled Tale of Quantum Entanglement

This work is supported by:

- The National Institute for Standards and Technology (NIST).
- The Mathematical Sciences Research Institute (MSRI).
- L-O-O-P The L-O-O-P Fund.

Talk based on the work of many people

- Benet
- Brassard
- Brylinski
- Nielsen
- Horodecki's
- Peres
- Plenio
- Popescu
- Schumacher
- Terhal
- Lomonaco
- Sudberry
- Wallach
- Meyer
- Jonathan
- Jozsa
- Linden

& Many Others
Over the 20-th century, the scientific community's view of Q.E. has dramatically changed.

- Initially, Q.E. was viewed as an unnecessary and unwanted ugly Wart on Quantum Mechanics.
  - EPR tried to surgically remove it.
  - Bell and Aspect showed that surgery cannot be performed w/o destroying the very life of physical reality.

Today, Q.E. is viewed as a useful resource in Q.M. It is viewed as a commodity to be traded and utilized.

Q.E. NASDAQ Exchange

Q.E. Savings & Loan
Quantum Entanglement & Quantum Computation

- Q.E. appears to be an important resource for quantum computation
- Many claim that it is Q.E. that somehow enables us to harness the vast parallelism of Quantum Mechanics.

What is Q.E.? ?

?? Questions ??

- How do we measure, quantify, and classify Q.E.?
- When is the Q.E. of two quantum systems the same? Different? When is the Q.E. of one quantum systems greater than another?
- Answers to the above questions are expected to have a profound impact on the development of Quantum Computation.
- Finding answers to these questions is intriguing.

Overview

- $L(n) = \otimes SU(2)$ Local Unitary Group
- $t(n) = \mathfrak{su}(2)$ Lie algebra of $L(n)$
- $U(2^n)$ Unitary Group
- $\mathfrak{u}(2^n)$ Lie algebra of $U(2^n)$

Overview (Cont.)

- $L(n) \times \mathfrak{u}(2^n) \rightarrow u(2^n)$
- We study Q.E. by lifting the above action to the induced infinitesimal action $\mathfrak{u}(n) \rightarrow \mathfrak{ve}(u(2^n))$.
- We use the induced infinitesimal action to quantify and classify Q.E. by constructing a complete set of Q.E. invariants.

Chapter 1

A Story of Two Qubits

- How Alice & Bob Learn to Live with Q.E. & Love it
Bob Pops the Big Question

What is Q.E. ???

Gee, I don’t know ???

A Story of Two Qubits

Our saga continues

• Alice & Bob visit the local Q.E. wholesale outlet
• They find on one of the shelves a box labeled as follows:

<table>
<thead>
<tr>
<th>Q.E., Inc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two Qubit Quantum System</td>
</tr>
<tr>
<td>2_AB</td>
</tr>
<tr>
<td>Consisting of Qubits</td>
</tr>
<tr>
<td>2_A, 2_B</td>
</tr>
</tbody>
</table>

The content label required by federal law reads

<table>
<thead>
<tr>
<th>Q Sys</th>
<th>Hil. Space</th>
<th>State</th>
<th>Unitary Transf.</th>
<th>State Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>2_AB</td>
<td>H_AB</td>
<td>ρ_AB</td>
<td>U(2)_A</td>
<td>u(2)_AB</td>
</tr>
<tr>
<td>2_A</td>
<td>H_A</td>
<td>ρ_A</td>
<td>U(2)_A</td>
<td>u(2)_A</td>
</tr>
<tr>
<td>2_B</td>
<td>H_B</td>
<td>ρ_B</td>
<td>U(2)_B</td>
<td>u(2)_B</td>
</tr>
</tbody>
</table>

ρ_AB = 
\[
\begin{pmatrix}
0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 \\
\end{pmatrix}
\]

(*) Not legally responsible for the effects of decoherence.

A Story of Two Qubits

• Alice & Bob purchase the two qubit system 2_AB.
• Outside the store, they open the box. Alice takes qubit 2_A, Bob takes qubit 2_B.
• They then separate with their respective qubits. Alice flies to the University of Melbourne. Bob flies to Vancouver, British Columbia to attend the University of British Columbia.

After they arrive, Alice has second thoughts about their purchase. She phones Bob, and rattles off in rapid succession the following two questions:

• Did we get our money’s worth of Q.E. ?
• How much Q.E. did we actually purchase ?

Bob immediately hangs up, and phones the Q.E. Consumer Protection agency, which refers him to the National Institute of Q.E. Standards & Technology (NIQEST) in Gaithersburg, MD.

After a long distance ($$$) phone conversation, NIQEST agrees to send Alice & Bob their STANDARDS Q.E. KIT.

The NIQEST rep. Takes a STANDARD Q.E. two qubit quantum system 2_A from self, places qubit 2_A’ together with a STANDARDS MANUAL in box A. He/She then takes qubit 2_B together with a STANDARDS MANUAL in Box B. He/She then sends the two boxes by overnight mail ($$$) respectively to Alice and Bob.

After receiving their two boxes, Alice and Bob open them, take out their respective qubits and read the enclosed manuals.
The STANDARDS MANUAL reads as follows:

Q.E. Yardstick 1. $2_{AB}$ and $2'_{A'B'}$ possess the same Q.E. if it is possible for Alice and Bob to use their own local reversible operations (either individually or collectively) to transform $2_{AB}$ and $2'_{A'B'}$ into one another. If this is possible, then $2_{AB}$ and $2'_{A'B'}$ are of the same entanglement type, written

$$2_{AB} \sim 2'_{loc A'B'}$$

The STANDARDS MANUAL reads as follows:

Q.E. Yardstick 2. $2_{AB}$ possesses more Q.E. than $2'_{A'B'}$ if it is possible for Alice and Bob (either individually or collectively) to apply their own reversible and irreversible local operations to their local qubits to transform $2_{AB}$ into $2'_{A'B'}$. In this case, we write

$$2_{AB} \geq 2'_{loc A'B'}$$

CAVEAT: Q.E. may be irrevocably lost if Q.E. Yardstick 2 is applied.

Summary

• **Question.** What type of Q.E. do Alice and Bob collectively possess?

• **Question.** Is the Q.E. of $2_{AB}$ the same as the Q.E. of $2'_{A'B'}$?

• **Question.** Is the Q.E. of $2_{AB}$ greater than the Q.E. of $2'_{A'B'}$?

Chapter 2

Definition of the Problem of Quantum Entanglement

• Let $Z_1, Z_2, \ldots, Z_n$ be $n$ qubits, and let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ denote their respective Hilbert spaces

• Let $Z = \{Z_1, Z_2, \ldots, Z_n\}$ be the global quant. sys. with Hilbert Space $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$

**Fundamental Problem of Q.E. (FPQE).** Let $\rho$ and $\rho'$ be density operators representing two different states of $Z$. Is it possible to move $Z$ from state $\rho$ to state $\rho'$ by applying only local moves?

**Question.** But what is a local move?
Local Moves

1) (Standard) local unitary transformations
\[ \otimes_{k=1}^n U_k \in \otimes_{k=1}^n U(\mathcal{H}_k) \]
For example, \( U_A \otimes I, I \otimes U_B, U_A \otimes U_B \)

2) Measurement of (Standard) local observables
\[ \otimes_{k=1}^n O_k \in \otimes_{k=1}^n \text{Observables}(\mathcal{H}_k) \]
For example, \( O_A \otimes I, I \otimes O_B, O_A \otimes O_B \)

The Group \( L(n) \) of Local Unitary Transformations

Definition. The group of local unitary transformations \( L(n) \) is the subgroup of \( U(2^n) \) defined by
\[ L(n) = \otimes^n SU(2) \]

Restricted FPQE (RFPQE): Given two density operators \( \rho \) and \( \rho' \) in the Lie algebra \( u(2^n) \), does there exist a local move \( U \in L(n) \) s.t.
\[ \text{Ad}_U(i\rho) = U(i\rho)U^{-1} = i\rho' \]

Convention. From this point on, we consider only the RFPQE. Thus, for the rest of the talk
Local Moves = \( L(n) \)

Set of Entanglement Classes \( u(2^n)/L(n) \)

Terminology

Definition. Two elements \( \rho \) and \( \rho' \) in \( u(2^n) \) are said to be locally equivalent, written
\[ i\rho \sim i\rho' \]
provided there exists a \( U \in L(n) \) such that
\[ i\rho' = \text{Ad}_U(i\rho) = U(i\rho)U^{-1} \]
The equivalence class
\[ [i\rho]_E = \{i\rho' : i\rho' \sim i\rho\} \]
is called an entanglement class (or orbit). Finally, let \( u(2^n)/L(n) \) denote the set of entanglement classes.

But What Is Q.E.? We will achieve the following two objectives:

Objective 1. Find the dimension of each entanglement class. Given \( i\rho \), find \( \text{Dim}([i\rho]_E) \)

Objective 2. Find a complete set of invariants which classify all entanglement classes. In other words, find a finite set \( \{f_1, f_2, ..., f_K\} \) of real valued functions on \( u(2^n) \) which distinguish all entanglement classes, i.e.,
\[ i\rho \sim i\rho' \iff f_k(i\rho) = f_k(i\rho') \quad \forall k \]
Chapter 3

Applications of Lie Group

Physical Perspective

<table>
<thead>
<tr>
<th>Physics</th>
<th>Math</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hilbert Space $\mathbb{H}$</td>
<td></td>
</tr>
<tr>
<td>Dimension $\dim(\mathbb{H}) = N$</td>
<td></td>
</tr>
<tr>
<td>Unitary Group $U(N)$</td>
<td></td>
</tr>
<tr>
<td>Lie Group $\mathfrak{u}(N)$</td>
<td></td>
</tr>
</tbody>
</table>

| Observables: $\rho$ | Observables: $i\hat{o}$ |
| $N \times N$ Hermitian Ops $A$ | $N \times N$ Skew Hermitian Ops $\hat{A}$ |
| $A^\dagger = A^T = \hat{A}$ | $(\hat{A})^\dagger = [\hat{A}]^T = -\hat{A}$ |
| where $\mathfrak{u}(N) = \text{Lie algebra of } U(N)$ |

| Dynamics via $U \in U(N)$ $\psi \mapsto U \psi$ $\rho \mapsto U \rho U^\dagger$ |
| where $Ad_j(\hat{\rho}) = U(\hat{\rho})U^\dagger$ is the Lie adjoint rep. |

Pauli Spin Matrices

\[
\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

We also denote the $2 \times 2$ identity matrix by

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Coordinate Chart for $u(2)$

\[
\pi: u(2) \to \mathbb{R}^4
\]

\[
i\rho \mapsto (x_0, x_1, x_2, x_3)
\]

Basis for the Lie Algebra $u(2)$

- A vector space basis for $u(2)$ is
  \[
  \left\{ \xi = \frac{1}{2} i \sigma_0, \xi = \frac{1}{2} i \sigma_1, \xi = \frac{1}{2} i \sigma_2, \xi = \frac{1}{2} i \sigma_3 \right\}
  \]

- If $i\rho \in u(2)$, then
  \[
  i\rho = x_0 \xi_0 + x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3
  = x_0 \xi_0 + x_3 \xi_3
  \]
  where
  \[
  \begin{cases}
  \begin{aligned}
  x_j &= -1 &\quad &\text{if } i\rho \text{ is a density op} \\
  \mathbf{x} &= (x_1, x_2, x_3) &\in &\mathbb{R}^3 \\
  \mathbf{\xi} &= (\xi_0, \xi_1, \xi_2, \xi_3) &\in &u(2) \times u(2) \times u(2)
  \end{aligned}
  \end{cases}
  \]

Basis for the Lie Algebra $u(2^3)$

- A basis for $u(2^3)$ is
  \[
  \left\{ \xi_0 = \frac{1}{2} i \sigma_0 \otimes \sigma_0, \xi_1 = \frac{1}{2} i \sigma_0 \otimes \sigma_1, \ldots, \xi_9 = \frac{1}{2} i \sigma_2 \otimes \sigma_2 \right\}
  \]
  \[
  = \left\{ \xi_0 = \frac{1}{2} i \sigma_j \otimes \sigma_k : j,k = 0,1,2,3 \right\}
  \]

- $i\rho = \sum_{j,k=0}^{3} x_{j,k} \xi_{j,k}$
  where $x_{00} = -1$ if $i\rho$ is a density operator
Coordinate Chart for $u(2^n)$

$$\pi: u(2^n) \rightarrow \mathbb{R}^{4^n}$$

$$i\rho \mapsto (x_{00,00}, x_{00,01}, \ldots, x_{33,33})$$

Basis for the Lie Algebra $u(2^n)$

A basis for $u(2^n)$ is:

$$\{\xi_{j_1 \cdots j_n} = -\frac{1}{2} i \bigotimes_{k=1}^n \sigma_{j_k} : j_1, j_2, \ldots, j_n = 0, 1, 2, 3\}$$

$$i\rho = \sum_{j_1 \cdots j_n} x_{j_1 \cdots j_n} \xi_{j_1 \cdots j_n}$$

Overview

$$u(N) = T_u U(N) = \text{Vec}_g (U(N)) = \text{Der} (\mathcal{C}^\infty U(N))$$

$\forall \in u(N)$ is simultaneously each of the following:

- An $N \times N$ skew Hermitian matrix
- A tangent vector to $U(N)$ at $I$
- A right invariant vector field on $U(N)$
- A derivation (directional derivative) on $C^\infty (U(N)) = \{ f: U(N) \rightarrow \mathbb{R} : f \text{ smooth} \}$

Also Recall

- In $U(2^n)$, there are $4^n$ independent directions to move in, namely

$$\{\xi_{j_1 \cdots j_n} = -\frac{1}{2} i \bigotimes_{k=1}^n \sigma_{j_k} : j_1, j_2, \ldots, j_n = 0, 1, 2, 3\}$$

- For example, from $g \in U(2)$ we could move from $g$ in the direction $\xi_2 = -\frac{1}{2} i \sigma_2$

by following the curve

$$\gamma(t) = e^{t\xi_2} g$$

from $t=0$ on.

The Special Unitary Group $SU(N)$

$$SU(N) = \{ U \in U(N) : \det (U) = 1 \}$$

$$su(N) = \{ v \in u(N) : \text{trace} (v) = 0 \}$$

- A basis for $su(2^n)$ is:

$$\{\xi_{j_1 \cdots j_n} = -\frac{1}{2} i \bigotimes_{k=1}^n \sigma_{j_k} : j_1 \in \{0, 1, 2, 3\} \forall k \}$$

Not all $j_k$ are zero.

- Therefore $\text{Dim} (SU(2^n)) = 4^n - 1$
Big Adjoint & little adjoint

* Big Adjoint $\text{Ad}$ for global dynamics of $\mathfrak{g}$
  
  \[ U(N) \times u(N) \xrightarrow{\text{Ad}} u(N) \]
  
  \[(U, i\rho) \mapsto \text{Ad}_U(i\rho) = U(i\rho)U^{-1} \]

* little adjoint $\text{ad}$ for infinitesimal dynamics of $\mathfrak{g}$
  
  \[ u(N) \times u(N) \xrightarrow{\text{ad}} u(N) \]
  
  \[(v, i\rho) \mapsto \text{ad}_v(i\rho) = [v, i\rho] \]

where \([v, i\rho] = v(i\rho) - (i\rho)v\)

The Special Orthogonal Group $\text{SO}(3)$

* The Lie group of rotations in $\mathbb{R}^3$ is
  
  \[ \text{SO}(3) = \{ A \in \text{GL}(3, \mathbb{R}) : A^T = A^{-1} \text{ and det}(A) = 1 \} \]

* Its Lie algebra is
  
  \[ \text{so}(3) = \{ A \in \text{Mat}(3, \mathbb{R}) : v^T = -v \text{ and trace}(v) = 0 \} \]

* And we have \text{Dirac Belt Trick}:
  
  \[ \begin{align*}
  u(2) & \xrightarrow{\text{ad}} \text{End}(u(2)) = \text{so}(3) \\
  \exp & \downarrow \quad \downarrow \exp \\
  U(2) & \xrightarrow{\text{ad}} \text{Aut}(u(3)) = \text{SO}(3)
  \end{align*} \]

The Lie algebra $\ell(n)$ of the Local Group $L(n)$

**Local Group**

\[ L(n) = \bigotimes_j SU(2) \subset U(2^n) \]

\[ L(n) \times u(2^n) \xrightarrow{\text{Ad}} u(2^n) \]

The Lie algebra $\ell(n)$ of $L(n)$ is the sub-Lie algebra of $u(2^n)$ generated by:

\[ \{ \xi_j, \xi_{k_1\ldots k_j} : k_j \in \{0,1,2,3\} \forall j, \text{ and where exactly one } k_j \neq 0 \} \]

Invariants of Quantum Entanglement

Chapter 4

The Exponential Map

\[ \exp : u(N) \to U(N) \]

\[ v \mapsto e^v = \sum_{k=0}^{\infty} \frac{v^k}{k!} \]

Moreover, \( \text{Ad}_{\exp(v)} = \exp(\text{ad}_v) \), i.e.,

\[ v \mapsto \text{ad}_v, \]

\[ u(N) \xrightarrow{\text{ad}} \text{End}(u(N)) \]

\[ \exp \downarrow \quad \downarrow \exp \]

\[ U(N) \xrightarrow{\text{ad}} \text{Aut}(u(N)) \]

\[ U \mapsto \text{Ad}_U \]

The Lie algebra $\ell(n)$ of the Local Group $L(n)$

For example,

* $\ell(1)$ is generated by \{ $\xi_1, \xi_2, \xi_3$ \}

* $\ell(2)$ by \{ $\xi_{01}, \xi_{02}, \xi_{03}, \xi_{10}, \xi_{20}, \xi_{30}$ \}

* $\ell(3)$ by \{ $\xi_{001}, \xi_{002}, \xi_{003}, \xi_{010}, \xi_{020}, \xi_{030}, \xi_{100}, \xi_{200}, \xi_{300}$ \}
What it is all about

The essence of the RFPQE is to understand the BIG Adjoint action of the local group \( L(n) \) on the Lie algebra \( u(2^n) \), namely, the action

\[
\text{Ad}_u \times i\rho \rightarrow u(2^n)
\]

What is a Q.E. invariant?

We seek invariants of the BIG Adjoint action of the group \( L(n) \) on the Lie algebra \( u(2^n) \).

A Complete Set of Invariants

If \( f \in C^\infty(u(2^n)) \), then

\[
\Rightarrow f(i\rho) = f(i\rho')
\]

However,

\[
f(i\rho) = f(i\rho') \Rightarrow i\rho \sim i\rho'
\]

In this case, we know nothing. The invariant is not enough to distinguish all Q.E. classes.

We seek enough Q.E. invariants to distinguish all Q.E. classes. Such a set of Q.E. invariants is called A Complete Set of Invariants.

How do we find Q.E. invariants?

We seek \( f : u(2^n) \rightarrow \mathbb{R} \) which are invariant under the BIG Adjoint action

\[
L(n) \times u(2^n) \rightarrow u(2^n)
\]

In other words, we seek \( f \) such that

\[
f(\text{Ad}_u(i\rho)) = f(i\rho), \forall i\rho \in u(2^n), \forall U \in L(N)
\]

Our Approach: Lift the problem to the Lie algebra \( \mathfrak{l}(n) \) where it becomes a linear problem.

The BIG Adjoint action of \( L(n) \) on \( u(2^n) \) induces an infinitesimal action

\[
\mathfrak{l}(n) \rightarrow \text{Vec}(u(2^n))
\]

Let \( v \in \mathfrak{l}(n) \). We define the vector field \( \Omega(v) \) by constructing the tangent vector \( \Omega(v)|_{i\rho} \) for each \( i\rho \in u(2^n) \). Let \( \gamma(t) \) be a smooth curve in \( u(2^n) \) defined by

\[
\gamma(t) = \text{Ad}_{\exp(v)}(i\rho)
\]

Then \( \gamma(t) \) passes through \( i\rho \) at time, \( t = 0 \). Define \( \Omega(v)|_{i\rho} \) to be the tangent vector to \( \gamma(t) \) at time \( t = 0 \).
The Infinitesimal Action

\[ \Omega(v)|_\rho = ad_v|_\rho \]

What is the meaning of the infinitesimal action?

Each \( \Omega(v)|_\rho \) is a direction in \( u(2^n) \) from which we can move w/o leaving the Q.E. class \( [i\rho]_\epsilon \).

Movement in all directions not in \( \text{Im} \left( \Omega(v)|_\rho \right) \) will force us to immediately leave \( [i\rho]_\epsilon \).

Objective 1. Find \( \text{Dim} \ [i\rho]_\epsilon \) (Achieved !!!)

Consider

\[ \ell(n) \xrightarrow{\alpha} \text{Im} \left( \Omega \right) \subset \text{Vec} \left( u(2^n) \right) \]

Then

\[ T_{\rho}[i\rho]_\epsilon = \text{Im} \left( \Omega \right)|_\rho \subset \text{Vec} \left( u(2^n) \right)|_\rho = T_{\rho}u(2^n) \]

Tangent Space to \([i\rho]_\epsilon \) at \( i\rho \)

Hence,

\[ \text{Dim} \ [i\rho]_\epsilon = \text{Dim} \ T_{\rho}[i\rho]_\epsilon = \text{Dim} \left( \text{Im} \Omega|_\rho \right) \]

The above approach leads to the following theorem:

Theorem. Let \( v_1, v_2, \ldots, v_{3n} \) be a vector space basis of the Lie algebra \( \ell(n) \). Then

\[ f \in \left( C^\infty u(2^n) \right)_{\ell(n)} \iff \Omega(v_j) \rightleftharpoons f = 0 \ \forall j \]

Hence, finding Q.E. invariants is equivalent to solving the above system of linear partial differential equations.

Obj. 2. Find a complete set of Q.E. invariants

\( \ell(n) \xrightarrow{\alpha} \text{Im}(\Omega) \subset \text{Vec} \left( u(2^n) \right) = \text{Der} \left( C^\infty u(2^n) \right) \)

Recall that \( \text{Im} \left( \Omega \right) \) consists of all directions in \( u(2^n) \) that we can move in w/o leaving a Q.E. class that we are presently in. If

\[ f \in \left( C^\infty u(2^n) \right)_{\ell(n)} \]

then \( f \) will not change if we move in any direction in \( \text{Im} \left( \Omega \right) \).

Chapter 5

Q.E. Invariants for \( n=1 \) Qubits
Q.E. Invariants for \( n=1 \) Qubits

- This is a trivial, but instructive case
- No Q.E.; but \( \exists \) many Q.E. classes

\[
\begin{align*}
\forall & \rho \in u(2) = \mathbb{R}^4 \\
\rho &= \sum_{j=0}^{3} \xi_j \sigma_j \\
(i \rho) &= \sum_{j=0}^{3} L_j \xi_j \\

\text{where} \\
L_0 &= 0 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
L_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
L_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
L_3 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{align*}
\]

are the infinitesimal generators of the Lie algebra \( \mathfrak{so}(3) \) of the special orthogonal group \( SO(3) \)

Thus, \( \text{Im}(\Omega) \) is spanned by

\[
\begin{align*}
\Omega(\xi_0) &= x_0 \frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \\
\Omega(\xi_1) &= x_0 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_0} - x_2 \frac{\partial}{\partial x_2} \\
\Omega(\xi_2) &= x_0 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_0}
\end{align*}
\]

Hence,

\[
\text{Dim}\left(\text{Im}(\Omega)\right) = \begin{cases} 2 & \text{if } \vec{x} \neq 0 \\ 0 & \text{if } \vec{x} = 0 \end{cases}
\]
So $\Omega(\xi)|_\rho$, $\Omega(\xi_2)|_\rho$, $\Omega(\xi_3)|_\rho$ span the tangent space to $[i\rho]_k$ at $i\rho$.

Moreover, $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$ is the normal vector field to $[i\rho]_k$.

Finally, a Complete Set of Invariants

The solution of

\[ \begin{align*}
\Omega(\xi_1)f = 0 \\
\Omega(\xi_2)f = 0 \\
\Omega(\xi_3)f = 0
\end{align*} \]

is

\[ f(i\rho) = \bar{x}^2 \]

Q.E. Invariants for $n=2$ Qubits

Chapter 6

For $L(2) = SU(2) \otimes SU(2)$

For $\ell(2) = su(2)^{\mathbb{C}} + su(2) = \mathbb{R}^4$

For $i\rho \in u(2H2) = \mathbb{R}^{16}$

where

\[ A + B = (A \otimes I) + (I \otimes B) \]

Kronecker Sum

\[ \{ \xi_{01}, \xi_{02}, \xi_{03}, \xi_{10}, \xi_{20}, \xi_{30} \} \]

For $\ell(2)$, basis = \{ $\xi_{01}, \xi_{02}, \xi_{03}, \xi_{10}, \xi_{20}, \xi_{30}$ \}

For $u(2^2)$, basis = \{ $\xi_{jk}$ : $j,k \in \{0,1,2,3\}$ \}

A basis for $\text{Vec}(u(2^2))$ is

\[ \left\{ \frac{\partial}{\partial x_{jk}} : j,k = 0,1,2,3 \right\} \]
So, 

\[
\text{ad}(\xi) = (0 \otimes L_1) \otimes I_4 = I_4 \otimes (0 \otimes L_1)
\]

A similar formula holds for \(\text{ad}(i\rho)\).

\[
\text{ad}(i\rho) = (0 \otimes L_1) \otimes I_4 = I_4 \otimes (0 \otimes L_1)
\]

\[\Omega(\xi) = \text{ad}(\xi)(i\rho) \cdot \nabla\]

where

\[
\nabla = \left(\frac{\partial}{\partial x^{0'}}, \frac{\partial}{\partial x^{1'}}, \frac{\partial}{\partial x^{2'}}, \frac{\partial}{\partial x^{3'}}\right)
\]

Hence,

\[
\Omega(\xi) = \text{ad}(\xi)(i\rho) \cdot \nabla
\]

\[\text{Im}(\Omega)\] is spanned by

\[
\begin{align*}
\Omega(\xi_0) &= x_0 \frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial x_1} + \sum_{i=2}^{3} (x_i \frac{\partial}{\partial x_i} - x_i' \frac{\partial}{\partial x_i'}) \\
\Omega(\xi_1) &= x_0 \frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial x_1} + \sum_{i=2}^{3} (x_i \frac{\partial}{\partial x_i} - x_i' \frac{\partial}{\partial x_i'}) \\
\Omega(\xi_2) &= x_0 \frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial x_1} + \sum_{i=2}^{3} (x_i \frac{\partial}{\partial x_i} - x_i' \frac{\partial}{\partial x_i'}) \\
\Omega(\xi_3) &= x_0 \frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial x_1} + \sum_{i=2}^{3} (x_i \frac{\partial}{\partial x_i} - x_i' \frac{\partial}{\partial x_i'})
\end{align*}
\]

It follows that, \(\text{Dim}(\text{Im}(\Omega)) = 6\) a.e.

In other words, almost all density operators \(i\rho\) belong to a 6 dimensional Q.E. class.

But ... there are some important exceptions.

Please note that all four 2-qubit Bell states lie in the same Q.E. class. Because of this, teleportation is possible.

Consider the Bell state

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ -1 \end{array}\right)
\]

So,

\[
\rho = |\psi\rangle \langle \psi| = \frac{1}{2} \left(\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right)
\]

In terms of the basis \(\{\xi_0\}\), this becomes

\[
i\rho = \frac{1}{2} (|\xi_0\rangle + (-1)|\xi_1\rangle + (+1)|\xi_2\rangle + (+1)|\xi_3\rangle)
\]

Thus, in this case \(\text{Im}(\Omega)|_\rho\) is spanned by

\[
\begin{align*}
\Omega(\xi_0) &= \frac{\partial}{\partial x_{02}} + \frac{\partial}{\partial x_{12}} \\
\Omega(\xi_1) &= -\frac{\partial}{\partial x_{13}} + \frac{\partial}{\partial x_{23}} \\
\Omega(\xi_2) &= -\frac{\partial}{\partial x_{21}} - \frac{\partial}{\partial x_{31}} \\
\Omega(\xi_3) &= \frac{\partial}{\partial x_{32}} + \frac{\partial}{\partial x_{23}}
\end{align*}
\]
Hence, the dimension of the Bell state entanglement class \([i\rho_{\text{Bell}}]_E\) is an exceptional 3, i.e.,

\[
\text{Dim}[i\rho_{\text{Bell}}]_E = \text{Dim}(1\Omega)|_{\rho_{\text{Bell}}} = 3
\]

A Complete Set of Q.E. Invariants for \(n=2\) Qubits

Let us use the above mentioned chart \(\Pi\) to make the identification

\[
i\rho = \begin{pmatrix}
x_{00} - x_{01} & \frac{x_{01} + x_{10}}{\sqrt{2}} \\
\frac{x_{01} - x_{10}}{\sqrt{2}} & x_{11}
\end{pmatrix}
\]

and let

\[
Z = x_{11} x_{00}^T
\]

The following set of 9 algebraically independent polynomial functions form a basic set of polynomial Q.E. invariants:

\[
\begin{array}{c}
\text{trace}(Z) \\
\text{trace}(Z^2) \\
\det(x_{xx}) \\
x_{00} x_{11} x_{11}^T - x_{00} x_{11}^T x_{11} + x_{11} x_{00}^T x_{11} - x_{11} x_{00} x_{11}^T \\
x_{00} x_{11} x_{11}^T x_{11} - x_{00} x_{11}^T x_{11} x_{11} + x_{11} x_{00}^T x_{11} x_{11} - x_{11} x_{00} x_{11}^T x_{11} \\
x_{00} x_{11} x_{11}^T x_{11} x_{11} - x_{00} x_{11}^T x_{11} x_{11} x_{11} + x_{11} x_{00}^T x_{11} x_{11} x_{11} - x_{11} x_{00} x_{11}^T x_{11} x_{11}
\end{array}
\]

But... they do not form a complete set of Q.E. invariants!!!

A tenth polynomial \(x_{00} \cdot \left(Zx_{00}^T \right) \times \left(Zx_{00}^T \right)\), which is algebraically dependent on the above polynomials is needed to determine the sign of the components of \(i\rho\) and to form a complete set of Q.E. invariants.

Chapter 7

Conclusion

The RFPQE “Lives” in the Following Mathematical Structure

\[\ell(n) \xrightarrow{\text{Infinitesimal Action}} \text{Vec}(u(2^n)) \xrightarrow{\text{Lift}} \text{Big Adjoint Action} \xrightarrow{\text{Ad}} L(n) \times u(2^n) \xrightarrow{\text{Ad}} u(2^n)\]

The General Problem of Finding a Complete Set of Q.E. Invariants for \(n\) Qubits

In the paper “An Entangled Tale of Quantum Entanglement,” it is shown how the general problem of finding a complete set of Q.E. invariants for \(n\) qubits can be reduced to the task of finding a complete set of solutions to the system of \(3n\) PDEs listed on the following slide:
For a Complete Set of Q.E. Invariants
Solve the Following System of 3n PDEs

\[
\begin{align*}
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0 \\
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0 \\
\vdots \\
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0
\end{align*}
\]

where, for example,

\[
x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}}
\]
denotes the \textbf{vector cross product} of the two vectors.

\[
x_{h_1, h_2, \ldots, h_{3n}} \quad \text{and} \quad \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}}
\]

The RFPQE is difficult for \( n \geq 3 \) qubits.

Some progress has been made for 3 and 4 qubits. For example, Meyer & Wallach have recently been able to count the number of entanglement classes for \( n=4 \) qubits.

For a complete set of Q.E. invariants, solve the following system of PDEs:

\[
\begin{align*}
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0 \\
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0 \\
\vdots \\
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0
\end{align*}
\]

The Big Adjoint action does not mathematically fully captures all of the physical phenomenon of Q.E.

There is more to Q.E. than the RFPQE.

Thinking inside or outside the box?

Considering classical communication (LOCC), the distillation of entangled states, and much more...

- The effects of classical communication (LOCC)
- The distillation of entangled states
- And much more...

For example, the mathematical model of the Big Adjoint action needs to be extended to capture such other physical effects of Q.E. as:

\[
\begin{align*}
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0 \\
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0 \\
\vdots \\
\sum_{h_1, h_2, \ldots, h_{3n}} x_{h_1, h_2, \ldots, h_{3n}} \times \frac{\partial f}{\partial x_{h_1, h_2, \ldots, h_{3n}}} &= 0
\end{align*}
\]

Thinking inside the box or outside the box?
Thinking Inside or Outside the Box?

\[ \triangle ABC \cong \triangle A'B'C' \]

3-D Physical Euclidean Geometry

The End


Quantum Computation and Information, Samuel J. Lomonaco, Jr. and Howard E. Brandt (editors), AMS CONM/305, (2002).