Quantum Knots

&

Mosaics

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This talk is based on the paper:


This talk was motivated by:

Lomonaco, Samuel J., Jr., The modern legacies of Thomson's atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 - 166.


Throughout this talk:

“Knot” means either a knot or a link
This talk was also motivated by:


Knots Naturally Arise in the Quantum World as Dynamical Processes

Examples of dynamical knots in quantum physics:
Knotted vortices
* In supercooled helium II
* In the Bose-Einstein Condensate
* In the Electron fluid found within the fractional quantum Hall effect

Reason for current intense interest:
A Natural Topological Obstruction to Decoherence

What Motivated This Talk?

Classical Vortices in Plasmas

Lomonaco, Samuel J., Jr., The modern legacies of Thomson's atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 - 166.

Objectives
* We seek to create a quantum system that simulates a closed knotted physical piece of rope.
* We seek to define a quantum knot in such a way as to represent the state of the knotted rope, i.e., the particular spatial configuration of the knot tied in the rope.
* We also seek to model the ways of moving the rope around (without cutting the rope, and without letting it pass through itself.)

Rules of the Game

Find a mathematical definition of a quantum knot that is
* Physically meaningful, i.e., physically implementable, and
* Simple enough to be workable and useable.

Aspirations
We would hope that this definition will be useful in modeling and predicting the behavior of knotted vortices that actually occur in quantum physics such as
* In supercooled helium II
* In the Bose-Einstein Condensate
* In the Electron fluid found within the fractional quantum Hall effect
Overview

Part 0. Quick Overview of Knot Theory

Part 1. Mosaic Knots
We reduce tame knot theory to a formal system of string manipulation rules, i.e., string rewriting systems.

Part 2. Quantum Knots
We then use mosaic knots to build a physically implementable definition of quantum knots.

Quick Overview to Knot Theory

Placement Problem: Knot Theory

\begin{itemize}
  \item Ambient space $= \mathbb{R}^3$
  \item Group $G = \text{AutoHomeo}(\mathbb{R}^3)$
\end{itemize}

Orientation Preserving

Def. $K \sim K_2$ if $g \in G$ s.t. $gK_1 = K_2$

Problem. When are two placements the same? $K_1 \sim K_2$?

What is a knot invariant?

Def. A knot invariant $I$ is a map $I: \text{Knots} \rightarrow \text{Mathematical Domain}$ that takes each knot $K$ to a mathematical object $I(K)$ such that

$K_1 \sim K_2 \Rightarrow I(K_1) = I(K_2)$

Consequently,

$I(K_1) \neq I(K_2) \Rightarrow K_1 \neq K_2$

The Jones polynomial is a knot invariant.

Knot Diagrams:
A fundamental tool in knot theory.

Knot Projections
Question: If we locally move the rope, what does its shadow (knot diagram) do???

In this case, we have not changed the topological type of the knot diagram.

This is a planar isotopy move denoted by R0.

When do two Knot diagrams represent the same or different knots?

Theorem (Reidemeister). Two knots diagrams represent the same knot iff one can be transformed into the other by a finite sequence of Reidemeister moves (and planar isotopy rules).
Mosaic Knots

Example of Application of Reidemeister Moves

Part 1

Mosaic Knots

Mosaic Tiles

Let $T^{(u)}$ denote the following set of 11 symbols, called mosaic (unoriented) tiles:

Please note that, up to rotation, there are exactly 5 tiles.

Definition of an $n$-Mosaic

An $n$-mosaic is an $n \times n$ matrix of tiles, with rows and columns indexed $0, 1, \ldots, n - 1$.

An example of a 4-mosaic

Tile Connection Points

A connection point of a tile is a midpoint of a tile edge which is also the endpoint of a curve drawn on the tile. For example,
Contiguous Tiles

Two tiles in a mosaic are said to be **contiguous** if they lie immediately next to each other in either the same *row* or the same *column*.

![Contiguous Tiles Diagram](image)

Suitably Connected Tiles

A tile in a mosaic is said to be **Suitably Connected** if all its connection points touch the connection points of contiguous tiles. For example,

![Suitably Connected Tiles Diagram](image)

Knot Mosaics

A **knot mosaic** is a mosaic with all tiles suitably connected. For example,

![Knot Mosaics Diagram](image)

Figure Eight Knot 5-Mosaic

![Figure Eight Knot 5-Mosaic](image)

Hopf Link 4-Mosaic

![Hopf Link 4-Mosaic](image)

Borromean Rings 6-Mosaic

![Borromean Rings 6-Mosaic](image)
**Notation**

\[ M^{(n)} = \text{Set of } n\text{-mosaics} \]

\[ K^{(n)} = \text{Subset of knot } n\text{-mosaics} \]

**Planar Isotopy Moves**

**Non-Deterministic Tiles**

We use the following tile symbols to denote one of two possible tiles:

For example, the tile \( \) denotes either or \( \).

**11 Planar Isotopy (PI) Moves on Mosaics**

It is understood that each of the above moves depicts all moves obtained by rotating the \( 2 \times 2 \) sub-mosaics by 0, 90, 180, or 270 degrees.

For example, represents each of the following 4 moves:

**Terminology: k-Submosaic Moves**

**Def.** A \( k \)-submosaic move on a mosaic \( M \) is a mosaic move that replaces one \( k \)-submosaic in \( M \) by another \( k \)-submosaic.

All of the PI moves are examples of \( 2 \)-submosaic moves. I.e., each PI move replaces a \( 2 \)-submosaic by another \( 2 \)-submosaic.

For example,
Planar Isotopy (PI) Moves on Mosaics

Each of the PI $2$-submosaic moves represents any one of the $(n-k+1)^2$ possible moves on an $n$-mosaic.

Reidemeister (R) Moves on Mosaics

Each PI move acts as a local transformation on an $n$-mosaic whenever its conditions are met. If its conditions are not met, it acts as the identity transformation.

Ergo, each PI move is a permutation of the set of all knot $n$-mosaics $K^{(n)}$.

In fact, each PI move, as a permutation, is a product of disjoint transpositions.

More Non-Deterministic Tiles

We also use the following tile symbols to denote one of two possible tiles:

For example, the tile \[ \square \] denotes either \[ \square \] or \[ \square \].

Synchronized Non-Deterministic Tiles

Nondeterministic tiles labeled by the same letter are synchronized:

\[ \square = \square \quad \square = \square \]
Reidemeister (R) Moves on Mosaics

Just like each PI move, each R move is a permutation of the set of all knot n-mosaics \( \mathcal{K}^{(n)} \).

In fact, each R move, as a permutation, is a product of disjoint transpositions.

The Ambient Group \( A(n) \)

We define the ambient isotopy group \( A(n) \) as the subgroup of the group of all permutations of the set \( \mathcal{K}^{(n)} \) generated by the all PI moves and all Reidemeister moves.

Knot Type

The Mosaic Injection \( \iota : M^{(n)} \to M^{(n+1)} \)

We define the mosaic injection \( \iota : M^{(n)} \to M^{(n+1)} \):

\[
M^{(n+1)}_{i,j} = \begin{cases} 
M^{(n)}_{i,j} & \text{if } 0 \leq i, j < n \\
0 & \text{otherwise}
\end{cases}
\]

Mosaic Knot Type

Def. Two n-mosaics \( M \) and \( M' \) are of the same knot n-type, written \( M \sim M' \), provided there exists an element of the ambient group \( A(n) \) that transforms \( M \) into \( M' \).

Two n-mosaics \( M \) and \( M' \) are of the same knot type if there exists a non-negative integer \( k \) such that \( \iota^k M \sim \iota^k M' \).
Oriented Mosaics and Oriented Knot Type

In like manner, we can use the following oriented tiles to construct oriented mosaics, oriented mosaic knots, and oriented knot type.

There are 29 oriented tiles, and 9 tiles up to rotation. Rotationally equivalent tiles have been grouped together.

The Hilbert Space $\mathcal{M}^{(n)}$ of n-mosaics

Let $\mathcal{H}$ be the 11 dimensional Hilbert space with orthonormal basis labeled by the tiles.

We define the Hilbert space $\mathcal{M}^{(n)}$ of n-mosaics as

$\mathcal{M}^{(n)} = \bigotimes_{k=0}^{n-1} \mathcal{H}$

This is the Hilbert space with induced orthonormal basis

$\left\{ \bigotimes_{k=0}^{n-1} |T_{\ell(k)}\rangle : 0 \leq \ell(k) < 11 \right\}$

The Hilbert Space $\mathcal{M}^{(n)}$ of n-mosaics

We identify each basis ket $\bigotimes_{k=0}^{n-1} |T_{\ell(k)}\rangle$ with a ket $|M\rangle$ labeled by an n-mosaic $M$ using row major order.

For example, in the 3-mosaic Hilbert space $\mathcal{M}^{(3)}$, the basis ket

$|T_0\rangle \otimes |T_0\rangle \otimes |T_0\rangle \otimes |T_0\rangle \otimes |T_0\rangle \otimes |T_0\rangle \otimes |T_0\rangle \otimes |T_0\rangle \otimes |T_0\rangle$ is identified with the 3-mosaic labeled ket

$T_2 \ T_4 \ T_9$

$T_6 \ T_2 \ T_4$

$T_8 \ T_4 \ T_6$
Identification via Row Major Order

Let $\mathbb{H}$ be the 11-dimensional Hilbert space with orthonormal basis labeled by the tiles $T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}$.

Construct Mosaic Space

$$\mathbb{H}^{\otimes n} = \bigotimes_{i=1}^{n} |T_{(i,j)}\rangle$$

Select Basis Element

Row Major Order

The Hilbert Space $\mathbb{H}^{(n)}$ of Quantum Knots

The Hilbert space $\mathbb{H}^{(n)}$ of quantum knots is defined as the sub-Hilbert space of $\mathbb{H}^{(n)}$ spanned by all orthonormal basis elements labeled by knot $n$-mosaics.

Quantum Knots

We define the Hilbert space $\mathbb{H}^{(n)}$ of $n$-mosaics as

$$\mathbb{H}^{(n)} = \bigotimes_{i=1}^{n} \mathbb{H}$$

This is the Hilbert space with induced orthonormal basis

$$\left\{ \bigotimes_{i=1}^{n-1} |T_{(i,j)}\rangle : 0 \leq \ell < n^2 \right\}$$

We identify each basis element $\bigotimes_{i=1}^{n-1} |T_{(i,j)}\rangle$ with the mosaic labeled ket $|M\rangle$ via the bijection $T_\ell \leftrightarrow M_{ij}$, where

$$\begin{cases} i = \lfloor \ell/n \rfloor \\ j = \ell - n \lfloor \ell/n \rfloor \end{cases} \quad \text{and} \quad \ell = ni + j$$

The Ambient Group $A(n)$ as a Unitary Group

We identify each element $g \in A(n)$ with the linear transformation defined by

$$\mathbb{H}^{(n)} \to \mathbb{H}^{(n)}

|K\rangle \mapsto |gK\rangle$$

An Example of the $A(n)$ Group Action

An Example of a Quantum Knot

Since each element $g \in A(n)$ is a permutation, it is a linear transformation that simply permutes basis elements.

Hence, under this identification, the ambient group $A(n)$ becomes a discrete group of unitary transforms on the Hilbert space $\mathbb{H}^{(n)}$. 

$$|K\rangle = \sqrt{2} \begin{pmatrix} \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \end{pmatrix}$$

An Example of the $A(n)$ Group Action

$$|K\rangle = \sqrt{2} \begin{pmatrix} \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \end{pmatrix}$$
The Quantum Knot System \( (\mathcal{K}^{(n)}, A(n)) \)

**Def.** A quantum knot system \( (\mathcal{K}^{(n)}, A(n)) \) is a quantum system having \( \mathcal{K}^{(n)} \) as its state space, and having the Ambient group \( A(n) \) as its set of accessible unitary transformations.

The states of quantum system \( (\mathcal{K}^{(n)}, A(n)) \) are quantum knots. The elements of the ambient group \( A(n) \) are quantum moves.

\[
(\mathcal{K}^{(n)}, A(n)) \to (\mathcal{K}^{(n)}, A(n)) \to (\mathcal{K}^{(n)}, A(n+1)) \to \ldots
\]

Physically Implementable

Choosing an integer \( n \) is analogous to choosing a length of rope. The longer the rope, the more knots that can be tied.

The parameters of the ambient group \( A(n) \) are the “knobs” one turns to spatially manipulate the quantum knot.

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**Quantum Knot Type**

**Def.** Two quantum knots \( |K_1\rangle \) and \( |K_2\rangle \) are of the same knot \( n \)-type, written

\[
|K_1\rangle \sim_n |K_2\rangle,
\]

provided there is an element \( g \in A(n) \) s.t.

\[
g|K_1\rangle = |K_2\rangle
\]

They are of the same knot type, written

\[
|K_1\rangle \sim |K_2\rangle,
\]

provided there is an integer \( m \geq 0 \) such that

\[
t^m |K_1\rangle \sim t^m |K_2\rangle
\]

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**Two Quantum Knots of the Same Knot Type**

\[
|K\rangle = \begin{pmatrix} \phantom{1} \end{pmatrix} + \begin{pmatrix} \phantom{1} \end{pmatrix}
\]

\[
R_1 |K\rangle = \begin{pmatrix} \phantom{1} \end{pmatrix} + \begin{pmatrix} \phantom{1} \end{pmatrix}
\]

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**Two Quantum Knots NOT of the Same Knot Type**

\[
|K_1\rangle = \begin{pmatrix} \phantom{1} \end{pmatrix}
\]

\[
|K_2\rangle = \begin{pmatrix} \phantom{1} \end{pmatrix} + \begin{pmatrix} \phantom{1} \end{pmatrix}
\]

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**Hamiltonians of the Generators of the Ambient Group**
Hamiltonians for $A(n)$

Each generator $g \in A(n)$ is the product of disjoint transpositions, i.e.,

$$ g = (K_{a_1} K_{a_2}) (K_{a_3} K_{a_4}) \cdots (K_{a_{i-1}} K_{a_i}) $$

Choose a permutation $\eta$ so that

$$ \eta^{-1} g \eta = (K_i, K_j) (K_i, K_j) \cdots (K_{i-1}, K_i) $$

Hence,

$$ \eta^{-1} g \eta = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \quad \text{where} \quad \sigma_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} $$

The Log of a Unitary Matrix

Let $U$ be an arbitrary finite $r \times r$ unitary matrix.

Then eigenvalues of $U$ all lie on the unit circle in the complex plane.

Moreover, there exists a unitary matrix $W$ which diagonalizes $U$, i.e., there exists a unitary matrix $W$ such that

$$ WUW^{-1} = \Delta(e^{i \theta_1}, e^{i \theta_2}, \ldots, e^{i \theta_r}) $$

where $e^{i \theta_1}, e^{i \theta_2}, \ldots, e^{i \theta_r}$ are the eigenvalues of $U$.

The Log of a Unitary Matrix

Since $e^4 = \sum_{m=0}^{\infty} A^m / (m!)^2$, we have

$$ e^{ln(U)} = W^{-1} e^{\Delta(ln(i \theta_1), \ldots, ln(i \theta_r))} W $$

$$ = W^{-1} e^{\Delta(ln(i \theta_1), \ldots, ln(i \theta_r))} W $$

$$ = W^{-1} \Delta(e^{i \theta_1}, \ldots, e^{i \theta_r}) W $$

$$ = W^{-1} \Delta(e^{i \theta_1 + 2\pi n_1}, \ldots, e^{i \theta_r + 2\pi n_r}) W $$

$$ = W^{-1} \Delta(e^{i \theta_1}, \ldots, e^{i \theta_r}) W = U $$

Hamiltonians for $A(n)$

Also, let $\sigma_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and note that

$$ \ln(\sigma_i) = \frac{i\pi}{2} (2s + 1)(\sigma_s - \sigma_i), \quad s \in \mathbb{Z} $$

For simplicity, we always choose the branch $s = 0$.

$$ H_s = -i \eta \ln(\eta^{-1} g \eta) \eta^{-1} $$

$$ = \frac{\pi}{2} \eta \begin{bmatrix} I_r \otimes (\sigma_s - \sigma_i) & 0 \\ 0 & 0 \end{bmatrix} \eta^{-1} $$

The Log of a Unitary Matrix

Then

$$ \ln(U) = W^{-1} \Delta(ln(e^{i \theta_1}), ln(e^{i \theta_2}), \ldots, ln(e^{i \theta_r})) W $$

Since $\ln(e^{i \theta_j}) = i \theta_j + 2\pi n_j$, where $n_j \in \mathbb{Z}$ is an arbitrary integer, we have

$$ \ln(U) = iW^{-1} \Delta(\theta_1 + 2\pi n_1, \theta_2 + 2\pi n_2, \ldots, \theta_r + 2\pi n_r) W $$

where $n_1, n_2, \ldots, n_r \in \mathbb{Z}$

Hamiltonians for $A(n)$

Using the Hamiltonian for the Reidemeister 2 move

$$ g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} $$

and the initial state

we have that the solution to Schrödinger’s equation for time $t$ is

$$ e^{\frac{it}{2}} \begin{bmatrix} \cos \left( \frac{t}{2} \right) & i \sin \left( \frac{t}{2} \right) \\ i \sin \left( \frac{t}{2} \right) & \cos \left( \frac{t}{2} \right) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \end{bmatrix} $$
**Some Miscellaneous Unitary Transformations Not in $A(n)$**

The crossing tunneling transformation

$$\tau_{ij} = \begin{pmatrix} i & j \end{pmatrix}$$

The mirror image transformation

$$\mu = \prod_{i,j=1}^{n} \begin{pmatrix} i & j \end{pmatrix}$$

**Misc. Transformations**

The hyperbolic transformation

$$\eta_{ij} = \begin{pmatrix} i \leftrightarrow j \end{pmatrix}$$

The elliptic transformation

$$\varepsilon_{ij} = \begin{pmatrix} i \leftrightarrow j \end{pmatrix}$$

**Observables which are Quantum Knot Invariants**

**Question.** What do we mean by a physically observable knot invariant?

Let $(\mathcal{K}^{(n)}, A(n))$ be a quantum knot system. Then a quantum observable $\Omega$ is a Hermitian operator on the Hilbert space $\mathcal{K}^{(n)}$.

**Question.** But which observables $\Omega$ are actually knot invariants?

**Def.** An observable $\Omega$ is an invariant of quantum knots provided $U\Omega U^{-1} = \Omega$ for all $U \in A(n)$.

**Observable Q. Knot Invariants**
**Observable Q. Knot Invariants**

**Question.** But how do we find quantum knot invariant observables?

**Theorem.** Let \( (\mathcal{K}^{(n)}, A(n)) \) be a quantum knot system, and let 
\[ \mathcal{K}^{(n)} = \bigoplus_{\ell} W_\ell \]
be a decomposition of the representation 
\[ A(n) \times \mathcal{K}^{(n)} \to \mathcal{K}^{(n)} \]
into irreducible representations.

Then, for each \( \ell \), the projection operator \( P_\ell \)
for the subspace \( W_\ell \) is a quantum knot observable.

**Future Directions & Open Questions**

- What is the structure of the ambient group \( A(n) \) and its direct limit \( A = \lim A(n) \)?
  Can one find a presentation of this group?
  Is \( A(n) \) a Coxeter group?

- Unlike classical knots, quantum knots can exhibit the non-classical behavior of quantum superposition and quantum entanglement. Are quantum and topological entanglement related to one another?
  If so, how?

**Theorem.** Let \( (\mathcal{K}^{(n)}, A(n)) \) be a quantum knot system, and let \( \Omega \) be an observable on \( \mathcal{K}^{(n)} \).

Let \( \text{St}(\Omega) \) be the stabilizer subgroup for \( \Omega \), i.e.,
\[ \text{St}(\Omega) = \{ U \in A(n) : U\Omega U^{-1} = \Omega \} \]

Then the observable
\[ \sum_{U \in \text{St}(\Omega)} U\Omega U^{-1} \]
is a quantum knot invariant, where the above sum is over a complete set of coset representatives of \( \text{St}(\Omega) \) in \( A(n) \).

**Future Directions & Open Questions**

- How does one find a quantum observable for the Jones polynomial?
  This would be a family of observables parameterized by points on the circle in the complex plane. Does this approach lead to an algorithmic improvement to the quantum algorithm created by Aharonov, Jones, and Landau?

- How does one create quantum knot observables that represent other knot invariants such as, for example, the Vassiliev invariants?
Future Directions & Open Questions

- What is gained by extending the definition of quantum knot observables to POVMs?
- What is gained by extending the definition of quantum knot observables to mixed ensembles?

Def. We define the mosaic number of a knot $k$ as the smallest integer $n$ for which $k$ is representable as a knot $n$-mosaic.
- The mosaic number of the trefoil is 4. In general, how does one compute the mosaic number of a knot? How does one find a quantum observable for the mosaic number?
- Is the mosaic number related to the crossing number of a knot?

Quantum Knot Tomography: Given many copies of the same quantum knot, find the most efficient set of measurements that will determine the quantum knot to a chosen tolerance $\varepsilon > 0$.

Quantum Braids: Use mosaics to define quantum braids. How are such quantum braids related to the work of Freedman, Kitaev, et al on anyons and topological quantum computing?

UMBC Quantum Knots Research Lab

We at UMBC are very proud of our new state of the art Quantum Knots Research Laboratory.
Weird !!!