Quantum Noise & Quantum Decoherence

Two Ways of Representing Quantum States

Ket $|\psi\rangle$ vs. Density Operator $\rho$

Example. We have seen pure ensembles, i.e., pure states. For example,

| Ket   | $|\psi\rangle$ |
|-------|-----------------|
| Prob  | 1               |

Problem. Certain other types of quantum states are difficult to represent in terms of kets $|\psi\rangle$

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If for example,
\[ |\psi\rangle = a|0\rangle + b|1\rangle \]
where
\[ |a|^2 + |b|^2 = 1 \]
then
\[ \rho = (a|0\rangle + b|1\rangle)(a\langle 0 + b\langle 1 |) \]

\[
\begin{pmatrix}
  a \\
  b
\end{pmatrix}
\begin{pmatrix}
  a^* & b^* \\
  a & b
\end{pmatrix}
\]

\[
\begin{pmatrix}
  |a|^2 & a b^* \\
  b a^* & |b|^2
\end{pmatrix}
\]

On the other hand,
\[ \rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1 | \]

\[
\begin{pmatrix}
  \frac{7}{8} & \frac{1}{8} \\
  \frac{1}{8} & \frac{1}{8}
\end{pmatrix}
\]

is the mixed ensemble

| Ket | |0\rangle | |1\rangle |
|-----|-------|-------|
| Prob | \frac{7}{8} | \frac{1}{8} |
**Question.** What happens when we ignore a component of a composite quantum system?

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We need one more tool, namely, the

**Partial Trace**

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Consider a quantum system $Q_{PE}$ which is a composite of the environment $Q_E$ and our principal quantum system $Q_P$. Then

$$
\rho_{PE} = \sum_{r,s,t,u} \lambda_{rstu} |\psi_r^P\rangle \langle \psi_r^P| |\psi_s^E\rangle \langle \psi_s^E| |\psi_t^P\rangle \langle \psi_t^P|
$$

$\downarrow$

Partial Trace $Tr_E \downarrow$ Ignore $Q_E$

$\downarrow$

$$
\rho_P = Tr_E(\rho_{PE}) = \sum_{r,s,t,u} \lambda_{rstu} \langle \psi_t^E | \psi_s^E \rangle |\psi_r^P\rangle \langle \psi_t^P|
$$

where we have performed the contraction

$$
|\psi_s^E\rangle \langle \psi_t^E| \rightarrow \langle \psi_t^E | \psi_s^E\rangle
$$

to obtain $\rho_P$ of $Q_P$.  

*We have “traced over the environment”*
Hence by ignoring the environment, we have created uncertainty!

\[
\begin{array}{ccc}
\text{Pure} & \text{Ignore } Q_E & \text{Mixed} \\
\text{Ensemble} & \Rightarrow & \text{Ensemble} \\
Q_{PE} & Tr_E & Q_P \\
\end{array}
\]

We have created uncertainty!

Purification

Surprisingly enough, we can also do the reverse

\[
\begin{array}{ccc}
\text{Mixed} & \text{Extend to} & \text{Pure} \\
\text{Ensemble} & \text{Higher Dim} & \text{Ensemble} \\
Q_P & \text{Q. Sys.} & Q_{PE} \\
\Rightarrow & \Rightarrow & \Rightarrow \\
\end{array}
\]

Purification

There are many different such extensions which all produce through unitary evolution the same behavior of $Q_P$. 
Methods for dealing with non-unitary evolution

Method 1. Operator Sum Representation

$\exists$ ops. $\{E_a\}$ such that

$$\rho(t) = E(t^{\text{init}}) = \sum_a E_a(\rho^{\text{init}}) E^a$$

Method 2. Purification

$|e\rangle \otimes |\psi\rangle \xrightarrow{\text{Entangle}} \sum_s |e_s\rangle \otimes M_s |\psi\rangle$

init., State | init., State
---|---
Environment | Environment

$\text{not. nec. orthogonal}$
$\text{not nec. normalized}$
Notation

Let $a = (a_1, a_2)$ and $b = (b_1, b_2) \in \{00, 01, 10, 11\}$

Then all 16 of the 2 qubit error patterns can be uniquely written as

$$\{ X^a Z^b \mid a, b \in \{00, 01, 10, 11\} \}$$

where

$$X^a Z^b = X_1^{a_1} X_2^{a_2} Z_1^{b_1} Z_2^{b_2}$$

For example,

$$X^{(0,1)} Z^{(1,1)} = X_1^0 X_2^1 Z_1^1 Z_2^1 = I_1 X_2 Z_1 Z_2$$

$$= (I \otimes I)(I \otimes X)(Z \otimes I)(I \otimes Z)$$

$$= Z \otimes XZ = Z \otimes Y = Z_1 Y_2$$

Types of Error Patterns

| $X^{(0,0)} Z^{(0,0)} = I \otimes I$ | 0-qubit Error Pattern |
| $X^{(1,0)} Z^{(1,0)} = Y \otimes I$ | 1-qubit Error Pattern |
| $X^{(1,1)} Z^{(1,0)} = Y \otimes X$ | 2-qubit Error Pattern |

In general

$$X^a Z^b \quad \text{Wt}(a \lor b)-\text{qubit Error Pattern}$$

where \( \lor \) denotes bitwise logical 'OR', and where \( \text{Wt}(a \lor b) \) denotes the Hamming weight of \( a \lor b \).