1. (20) Let $u \in \mathbb{R}^n$ such that $\|u\| = 1$. Consider the matrix $R = I - 2uu^T$. True or False? If $Rx = x$, then $u^Tx = 0$.

**Solution.** Note that $\alpha u = R(\alpha u)$ yields

$$\alpha u = \alpha u - 2\alpha uu^Tu = -\alpha u,$$

and $\alpha = 0$.

If $Rx = 0$, then $x = 2u\left(u^Tx\right)$, and $u^Tx = 0$.

Mark one and explain.

- True
- False

2. (20) Let $U$ be a unitary matrix, and $x, y$ are two nonzero vectors. True or False?

$$\cos \theta_{x,y} = \cos \theta_{Ux,Uy} \left( \cos \theta_{x,y} \text{ is defined by } \frac{x^Ty}{\|x\|\|y\|} \right).$$

**Solution.** Since $(Ux)^TUy = x^TU^TUy = x^Ty$, one has $\cos \theta_{x,y} = \cos \theta_{Ux,Uy}$.

Mark one and explain.

- True
- False

3. (20) Let $T = I - uu^T$ with $\|u\| = 1$. True or False? $T$ is invertible.

**Solution.** $Tu = (I - uu^T)u = u - uu^Tu = 0$.

Mark one and explain.

- True
- False

4. (20) Let $A$ be an $n \times n$ matrix such that $A^2 = I$, and $A \neq I$. True or False? $A + I$ is invertible. (Hint: look at $(A + I)(A - I)$. *)
Solution. Assume that $A + I$ is invertible.

$$(A + I)(A - I) = A^2 - A + A - I = 0,$$
and $A - I = 0$, i.e. $A = I$.

Mark one and explain.
- True
- False

5. (20) Let $S$ be the subspace of all symmetric matrices (i.e. $S = \{A : A \in \mathbb{R}^{n \times n}, \text{and } A = A^T\}$), and $K$ be the subspace of all skew-symmetric matrices (i.e. $K = \{A : A \in \mathbb{R}^{n \times n}, \text{and } A = -A^T\}$),

(a) (10) True or False? $S \cap K = \{0\}$.

Solution. If $A = A^T = -A^T$, then $2A^T = 0$, and $A = 0$.

Mark one and explain.
- True
- False

(b) (10) Find $\dim S$ and $\dim K$.

Solution. Let $I_{ij}$ be a matrix with $(ij)$ entry 1, and all other entries 0. The set of $\frac{n^2 + n}{2}$ matrices

$$\{I_{ij} + I_{ji}, i \neq j, \text{and } I_{ii}\}$$
forms a basis for $S$. The set of $\frac{n^2 - n}{2}$ matrices

$$\{I_{ij} - I_{ji}, i \neq j\}$$
forms a basis for $K$.

$\dim S =$
$\dim K =$

6. (20) True or False? If $P$ is a projector and $P \neq I$, then index $P = 1$.

Solution. $P^2 = P$, and $P \neq I = P^0$.

Mark one and explain.
- True
- False