MATH 430
quiz #2, 04/04/13
Total 100
Solutions

Show all work legibly.

1. (20) True or False? If $AB$ is a product of square matrices, and $A$ is singular, then $AB$ is singular.

Solution. Since $A$ is singular, the rows of $A$ are linearly dependent, and there is $x \neq 0$ such that $0 = x^T A$. This implies $0 = x^T (AB)$, hence the rows of $AB$ are linearly dependent, and $AB$ is singular.

Mark one and explain.  
- True  
- False

2. (20) Let $A$ be an $n \times n$ matrix. True or False? If $R(A) \cap N(A) = \{0\}$, then $N(A) = N(A^2)$.

Solution. If $Ax = 0$, then $A(Ax) = 0$. This shows that $N(A) \subseteq N(A^2)$. Let $x \in N(A^2) - N(A)$. Denote $Ax$ by $y$. If $A = [a_1, \ldots, a_n]$, then $y = x_1 a_1 + \ldots + x_n a_n \in R(A)$, and $y = Ax \neq 0$. On the other hand $Ay = AAx = 0$, $0 \neq y \in R(A) \cap N(A)$. This contradicts the condition $R(A) \cap N(A) = \{0\}$, and completes the proof.

Mark one and explain.  
- True  
- False

3. (50) Let $A$ be an $n \times n$ matrix such that $A^2 = A$. True or False?

(a) (10) True or False? $\mathbb{R}^n = R(A) + R(I - A)$

Solution. Every $x \in \mathbb{R}^n$ can be written as $x = Ax + (I - A)x$.

Mark one and explain.  
- True  
- False

(b) (20) True or False? $R(A) \cap N(A) = \{0\}$

Solution. Note that $A(I - A) = (I - A)A = 0$. Next note that if $z \in R(A) \cap R(I - A)$, then $z = 0$. Indeed 

$$z = Ax \implies Az = A^2 x \implies Az = z.$$
At the same time if
\[ z = (I - A)y, \text{ then } z = Az = A(I - A)y = 0. \]

Mark one and explain.
- True
- False

(c) True or False? \( \text{rank}(A) + \text{rank}(I - A) = n. \)

Solution. If \( \{a_1, \ldots, a_p\} \) is a basis for \( R(A) \), and \( \{b_1, \ldots, b_q\} \) is a basis for \( R(I - A) \), then the vector set \( \{a_1, \ldots, a_p, b_1, \ldots, b_q\} \) spans \( \mathbb{R}^n \) (see (a) above), and linearly independent (see (b) above), hence \( p + q = \text{rank}(A) + \text{rank}(I - A) = n. \)

Mark one and explain.
- True
- False

4. (30) Let \( T \) be a linear operator that maps \( v \in \mathbb{R}^2 \) into its orthogonal projection on the line \( x = y. \)

(a) (15) Find the matrix representation \( A \) of \( T \) in the basis \( \{v_1, v_2\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \)

Solution. \( T(v_1) = v_1, T(v_2) = 0, \text{ hence } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \)

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

(b) (15) Find the matrix representation \( A \) of \( T \) in the basis \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \)

\( T(v_1) = \frac{1}{2}v_1 + \frac{1}{2}v_2, \text{ and } T(v_2) = \frac{1}{2}v_1 + \frac{1}{2}v_2, \text{ hence } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \)

\[ A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \]