1. (20) True or False? If $A$ is a matrix, $Ax_i = y_i$, $i = 1, \ldots, n$ and $\{y_1, \ldots, y_n\}$ is a linearly independent set, then $\{x_1, \ldots, x_n\}$ is a linearly independent set.

**Solution.** Assume $\{x_1, \ldots, x_n\}$ is a linearly dependent set. Let $c_1, \ldots, c_n$ be $n$ scalars not all zeros ($c_1^2 + \ldots + c_n^2 > 0$) such that $0 = c_1 x_1 + \ldots + c_n x_n$. This equation yields

$$0 = A(c_1 x_1 + \ldots + c_n x_n) = c_1 y_1 + \ldots + c_n y_n.$$ 

This contradicts linear independence of $\{y_1, \ldots, y_n\}$, and concludes the proof.

2. (20) If $A$ and $B$ are two $n \times n$ upper triangular matrices compute $\text{tr}(AB)$.

**Solution.** Note that $AB = A_{11}B_{11} + A_{12}B_{21} + \ldots + A_{nn}B_{n1}$, and $\text{tr}(AB) = \text{tr}(A_{11}B_{11} + A_{12}B_{21} + \ldots + A_{nn}B_{n1})$.

Finally $\text{tr}(A_{ii}B_{ii}) = a_{ii}b_{ii}$, and $\text{tr}(AB) = a_{11}b_{11} + a_{22}b_{22} + \ldots + a_{nn}b_{nn}$.

3. (20) For $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ solve the matrix equation $AX -XA = A$.

**Solution.** Assume that $B$ solves the equation. Note that $5 = \text{tr}A = \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0$. This contradiction shows that the equation has no solutions.

4. (20) True or False? If $A$ is an $n \times n$ is a matrix such that $\sum_{j=1}^{n} a_{ij} = 1$ for each $i = 1, 2, \ldots, n$, then $A^{-1}$ does not exist.

**Solution.** The example $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ shows that $A$ may be invertible.

5. (20) True or False? If $A$ is an $n \times n$ matrix such that each column of $A$ adds up to 0, then $A^{-1}$ does not exist.

**Solution.** If $e = (1, \ldots, 1)^T$ (the vector whose entries are all 1), then $e^TA = 0$. This shows that rank $A < n$, hence $A^{-1}$ does not exist.

6. (20) True or False? If $A$ is an $n \times n$ matrix and $v \in \mathbb{R}^n$, then rank $(A + vv^T) \leq \text{rank } A + 1$.

**Solution.** Due to Example 4.4.8 in the textbook one has rank $(A + B) \leq \text{rank } A + \text{rank } B$. Since rank $vv^T = 1$ the result follows.