Show all work legibly.

1. (20) True or False? If $A = [a_1, \ldots, a_m]$ is an $n \times m$ matrix and $\forall x \in \mathbb{R}^m$ one has $Ax = 0$, then $A = 0$.

**Solution.** Note that $Ax = x_1a_1 + \ldots + x_ma_m$. If $x_i$ is a vector with $i^{th}$ coordinate 1, and all other coordinates 0, then $Ax_i = a_i = 0$ (i.e. if $x_1 = (1, 0, \ldots, 0)^T$, then $Ax_1 = a_1 = 0$).

2. (20) Let $\{a_1, \ldots, a_n\}$ be a linearly independent vector set. True or False? If the entries $b_{ij}$ of the matrix $B$ are defined by $b_{ij} = a_i^T a_j$, then $\det B \neq 0$ (i.e. rank $B = n$, and $n$ columns of $B$ are linearly independent).

**Solution.** Let $A = [a_1, \ldots, a_n]$. Note that $B = A^T A$, hence if $Bx = 0$, then $0 = Bx = x^T A^T A x = \|Ax\|^2$ and $Ax = 0$.

Since $\{a_1, \ldots, a_n\}$ is a linearly independent set $x = 0$, and $\det B \neq 0$.

3. (20) True or False? If $A = [a_1, \ldots, a_m]$, then $AA^T = a_1a_1^T + \ldots + a_ma_m^T$.

**Solution.** Since $A = [a_1, 0, \ldots, 0] + \ldots + [0, \ldots, 0, a_n]$ a direct computation shows that $AA^T = a_1a_1^T + \ldots + a_ma_m^T$.

4. (20) Let $u, v \in \mathbb{R}^n$ so that $v^T u \neq 1$. For the matrices $I - uv^T$ and $I - \frac{uv^T}{v^T u - 1}$ compute the product $(I - uv^T) \left( I - \frac{uv^T}{v^T u - 1} \right)$.

**Solution.** $(I - uv^T) \left( I - \frac{uv^T}{v^T u - 1} \right) = I$

5. (20) True or False? $\text{tr}(AB) = \text{tr}(BA)$.

**Solution.** Let $A = (a_{ij})$, and $B = (b_{ij})$.

$$\text{tr}(AB) = \sum_{ij} a_{ij}b_{ji} = \sum_{ij} b_{ij}a_{ji} = \text{tr}(BA).$$
6. (20) True of False? If \( A = A^T \), then the eigenvalues \( \lambda_i \) are all real, and \( \mathbf{v}_i^T \mathbf{v}_j = \delta_{ij} \).

**Solution.**

Suppose that \( \lambda = \alpha + i\beta \), and \( \mathbf{x} \) is an eigenvector corresponding to \( \lambda \), i.e.

\[
A \mathbf{x} = \lambda \mathbf{x}, \quad \mathbf{x} \neq 0.
\]  

(0.1)

Since \( \lambda = \alpha + i\beta \) is a complex number the vector \( \mathbf{x} = \mathbf{v} + iw \), where \( \mathbf{v} \) and \( \mathbf{w} \) are real vectors of dimension \( n \). The condition \( \mathbf{x} \neq 0 \) implies \( \|\mathbf{x}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 > 0 \). Separating the real and imaginary parts in (0.1) we get

\[
A\mathbf{v} = \alpha \mathbf{v} - \beta \mathbf{w}, \quad \text{and} \quad A\mathbf{w} = \beta \mathbf{v} + \alpha \mathbf{w},
\]

and the left multiplication of the first equation by \( \mathbf{w}^T \) and the left multiplication of the second equation by \( \mathbf{v}^T \) yield

\[
\mathbf{w}^T A\mathbf{v} = \alpha \mathbf{w}^T \mathbf{v} - \beta \mathbf{w}^T \mathbf{w}, \quad \text{and} \quad \mathbf{v}^T A\mathbf{w} = \beta \mathbf{v}^T \mathbf{v} + \alpha \mathbf{v}^T \mathbf{w}.
\]

Since \( \mathbf{v}^T A\mathbf{w} - \mathbf{w}^T A\mathbf{v} = 0 \) one has \( \beta \left[ \mathbf{v}^T \mathbf{v} + \mathbf{w}^T \mathbf{w} \right] = \beta \|\mathbf{x}\|^2 = 0 \). This implies \( \beta = 0 \), and shows that \( \lambda \) is real.

Any two nonzero vectors \( \mathbf{v}_i, \mathbf{v}_j \) are eigenvectors of the identity matrix, hence \( \mathbf{v}_i^T \mathbf{v}_j \) does not have to be equal to \( \delta_{ij} \).

7. (20) True of False? If \( A = A^T \), then \( \text{tr}(A) = \lambda_1 + \ldots + \lambda_n \).

**Solution.** If \( A = A^T \), then \( A = V^T D V \), where \( D = \text{diag} \{ \lambda_1, \ldots, \lambda_n \} \), \( V^T V = VV^T = I \). Note that

\[
\text{tr}(A) = \text{tr}(V^T D V) = \text{tr}(D V V^T) = \text{tr}(D) = \lambda_1 + \ldots + \lambda_n.
\]

8. (20) If \( A^{-1} \) exits, then \( A \mathbf{x} = \lambda \mathbf{x} \) yields \( \lambda \neq 0 \).

**Solution.** If \( \mathbf{x} = 0 \), then \( 0 = A \mathbf{x} = 0 \mathbf{x} \).

9. (20) True of False? If \( S = -S^T \) (is skew–symmetric), then \( I + S \) is nonsingular.

**Solution.** Note that

\[
(I + S)(I + S)^T = (I + S)(I - S) = I + S^T S.
\]
If \((I + S)^T x = 0\), then

\[
0 = x^T (I + S)(I + S)^T x = x^T \left( I + SS^T \right) x = \|x\|^2 + \|S^T x\|^2,
\]

and \(x = 0\). This yields nonsingularity of \((I + S)(I + S)^T\). Hence \(I + S\) is nonsingular.

10. (20) True of False? If \(A\) and \(B\) are nonsingular, and \(AB = BA\), then \(AB^{-1} = B^{-1}A\).

Solution.

\[
AB = BA \implies A = BAB^{-1} \implies B^{-1}A = AB^{-1}.
\]