1. (20) If $A$ is an $n \times n$ matrix, and $\{b_1, \ldots, b_k\} \subset \mathbb{R}^n$ so that the vector set $\{Ab_1, \ldots, Ab_k\}$ is linearly independent. True or False? The vector set $\{b_1, \ldots, b_k\}$ is linearly independent.

**Solution.** Let $\alpha_1, \ldots, \alpha_k$ be a set of scalars so that $0 = \alpha_1 b_1 + \ldots + \alpha_k b_k$. Then

$$0 = A(\alpha_1 b_1 + \ldots + \alpha_k b_k) = \alpha_1 Ab_1 + \ldots + \alpha_k Ab_k.$$ 

Due to linear independence of the vector set $\{Ab_1, \ldots, Ab_k\}$ one has $0 = \alpha_1 = \ldots = \alpha_k$. This yields linear independence of the set $\{b_1, \ldots, b_k\}$.

2. (20) Let $A$ be an $n \times n$ matrix. True or False? If $\text{Null } A^k = \text{Null } A^{k+1}$, then $\text{Null } A^{k+1} = \text{Null } A^{k+2}$.

**Solution.** Let $x \in \text{Null } A^{k+2}$, i.e., $0 = A^{k+2}x = A^{k+1}(Ax)$. This shows that $Ax \in \text{Null } A^{k+1} = \text{Null } A^k$, and $0 = A^k(Ax) = A^{k+1}x$. Hence $\text{Null } A^{k+2} \subseteq \text{Null } A^{k+1}$.

3. (20) Let $A$ be an $n \times m$ matrix. True or False? $\dim \text{Null } A + \dim \text{Col } A = m$.

**Solution.** $\dim \text{Col } A$ is the number of pivot positions in an echelon form of $A$. $\dim \text{Null } A$ is a number of columns of $A$ that are not pivot columns. The sum of the two numbers is the total number of columns in $A$ which is $m$.

4. (20) Let

$$A = \begin{bmatrix} 1 & 0 & 7 & 2 & 5 \\ 2 & 1 & 15 & 0 & 20 \end{bmatrix}.$$ 

Find $\dim \text{Null } A$.

**Solution.** Since $\dim \text{Col } A = 2$, one has $\dim \text{Null } A = 5 - 2 = 3$.

5. (20) Let $L$ be a line in $\mathbb{R}^n$ defined by

$$L = \{x + ty, \ -\infty < t < \infty\} \text{ with } ||y|| = 1.$$ 

For a vector $a \in \mathbb{R}^n$ find $p$ the orthogonal projection of $a$ on $L$. 

Solution. Since $p \in L$ one has $p = x + \alpha y$. Note that the vector $a - p$ is orthogonal to the direction vector $y$, hence

$$0 = (a - x - \alpha y)^T y = (a - x)^T y - \alpha y^T y, \text{ and } \alpha = (a - x)^T y.$$ 

Finally $p = x + y(a - x)^T y$.

6. (20) True or False? If $A = [a_1, \ldots, a_m]$, then $AA^T = a_1 a_1^T + \ldots + a_m a_m^T$.

Solution. Since $A = [a_1, 0, \ldots, 0] + \ldots + [0, \ldots, 0, a_n]$ a direct computation shows that $AA^T = a_1 a_1^T + \ldots + a_m a_m^T$. 