Show all work legibly.

1. (20) Let \( A = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} \) be an \( n \times m \) matrix, and \( a_i \in \text{Nul} \, A, \, i = 1, \ldots, n \). What is the maximal possible rank of \( A \)?

**Solution.** For each \( i, \ldots, n \) one has \( 0 = a_i^T a_i \), hence \( A = 0 \). the maximal possible rank of \( A \) is:

2. (20) Let \( A = [A_1, A_2] \) be an \( n \times n \) matrix such that \( \dim R(A_1) + \dim R(A_2) = n \). True of False? \( A \) is invertible.

**Solution.** Let \( A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), and \( A_2 = A_1 \).

Mark one and explain.

\( \circ \) True \( \square \) False

3. (20) Let \( A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \). True of False? \( N(A) = N(A_1) \cap N(A_2) \).

**Solution.** If \( \mathbf{v} \in N(A) \), then \( \mathbf{v} \in N(A_1) \), and \( \mathbf{v} \in N(A_2) \), hence \( N(A) \subseteq N(A_1) \cap N(A_2) \). If \( \mathbf{v} \in N(A_1) \cap N(A_2) \), then \( \mathbf{v} \) is perpendicular to every row of \( A \), hence \( \mathbf{v} \in N(A) \), and \( N(A_1) \cap N(A_2) \subseteq N(A) \).

Mark one and explain.

\( \circ \) True \( \square \) False

4. (20) Let \( A \) be a matrix such that \( \dim N(A) = n - 1 \). True or False? \( A \) is a rank one matrix (i.e., \( A = u v^T \)).

**Solution.** Let \( \{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\} \) be a basis for \( N(A) \). Select \( \mathbf{v} \in N(A)^\perp \) with \( \mathbf{v}^T \mathbf{v} = 1 \). Note that the vector set \( \{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{v}\} \) is linearly independent, hence a basis for \( \mathbb{R}^n \). If \( \mathbf{u} = A \mathbf{v} \), then

\[ 0 = A \mathbf{v}_i = u \mathbf{v}^T \mathbf{v}_i, \quad i = 1, \ldots, n - 1; \text{ and } \mathbf{u} = A \mathbf{v} = u \mathbf{v}^T \mathbf{v} \]
Hence $A = uv^T$.

Mark one and explain.

- True  □  False

5. (40) Let $u_1, u_2, v_1, v_2 \in \mathbb{R}^n$, and $A$ is a sum of two matrices of rank one: $A = u_1v_1^T + u_2v_2^T$.

(a) (20) Suppose the vector set $\{v_1, v_2\}$ is linearly independent, but $u_1$ is a nonzero vector proportional to $u_2$ (i.e. $u_1 = cu_2$). Find dim $\text{Col } A$ and a basis $B$ for $\text{Col } A$.

Solution. $A = u_1v_1^T + u_2v_2^T = u_2(cv_1)^T + u_2v_2^T = u_2( cv_1 + v_2)^T$. Hence $A$ is a rank one matrix, and $B = \{u_2\}$.

$\dim \text{Col } A = B = \{u_2\}$.

(b) (20) Suppose the vector sets $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are linearly independent. Find dim $\text{Col } A$ and a basis $B$ for $\text{Col } A$.

Solution. If $y \in \text{Col } A$, then $y = Ax = u_1(v_1^T x) + u_2(v_2^T x) \in \text{span } \{u_1, u_2\}$, hence $\dim \text{Col } A \leq 2$. Let $w_1$ be a vector such that $v_1^T w_1 = 0$, and $v_2^T w_1 = 1$. Let $w_2$ be a vector such that $v_1^T w_2 = 1$, and $v_2^T w_2 = 0$. Note that $A(c_1w_1 + c_2w_2) = c_1u_2 + c_2u_1$. Hence $\text{Col } A = \text{span } \{u_1, u_2\}$, and dim $\text{Col } A = 2$.

$\dim \text{Col } A = B = \{u_1, u_2\}$. 