Show all work legibly.

1. (20) Let $P$ be a set of nonnegative real numbers that is bounded above. Let $S = \{s : s^2 \in P\}$. True or False? $S$ is a bounded set.

Solution. Since $P$ is bounded there is a positive number $M$ such that for each $p \in P$ one has

$$p \leq M.$$  

(1)

Assume that $S$ is not bounded. There exists $s \in S$ such that $\sqrt{M} < |s|$. Keeping in mind that $s^2 = p$ for some $p \in P$ we get

$$\sqrt{M} < |s|, \; \text{and} \; M < |s|^2 = s^2 = p.$$ 

This contradicts (1), shows that the assumption is false, and completes the proof.

Mark one and explain.

- True  
- False

2. (20) Find $\lim_{n \to \infty} \sqrt{n + 1} / n^2 + 1$.

Solution. When $n \geq 2$

$$0 < \frac{\sqrt{n + 1}}{n^2 + 1} < \frac{\sqrt{n + 1}}{n^2} < \frac{1}{n} \sqrt{n + 1} < \frac{1}{n}.$$ 

$$\lim_{n \to \infty} \frac{\sqrt{n + 1}}{n^2 + 1} = 0.$$ 

3. (20) Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$. True or False? The sequence $\{a_n\}$ converges.

Solution. There exists $N > 0$ such that for each $n \geq N$ one has $\frac{a_{n+1}}{a_n} > 2$. Hence $a_{N+k} > 2^k a_N$. This shows that the sequence is not bounded, hence is not convergent.

Mark one and explain.

- True  
- False

4. (30) Let $a_n = \frac{1}{n + 1} + \frac{1}{n + 2} + \ldots + \frac{1}{2n}$. 


(a) (20) True or False? The sequence \( \{a_n\} \) is bounded.

Solution.

\[
0 < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} < \frac{1}{n+1} + \frac{1}{n+1} + \ldots + \frac{1}{n+1} = \frac{n}{n+1} < 1.
\]

Mark one and explain.

True \quad False

(b) (20) True or False? The sequence \( \{a_n\} \) is increasing.

Solution.

\[
\begin{align*}
a_{n+1} - a_n &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \left[ \frac{1}{2n+1} - \frac{1}{2n+2} \right] + \left[ \frac{1}{2n+2} - \frac{1}{2n+2} \right] > 0.
\end{align*}
\]

Mark one and explain.

True \quad False

(c) (20) True or False? The sequence \( \{a_n\} \) converges.

Solution. Bounded and increasing sequence (see (a) and (b) above) converges.

Mark one and explain.

True \quad False