MATH 301 SOLUTIONS 9, KOGAN SPRING 2013

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SECTION 3.6

2a. Let $x_n = y_n = n$. Clearly, $x_n$ and $y_n$ are properly divergent. But $\frac{x_n}{y_n} = 1$, which is obviously convergent to 1.

2b. Let $x_n = n^2$ and $y_n = n$. Then $\frac{x_n}{y_n} = n$, which is properly divergent.

7a. Since $\lim \frac{x_n}{y_n} = 0$, for $\epsilon > 0$, there exists $N$ such that for $n \geq N$,

$$0 \leq \frac{x_n}{y_n} \leq \epsilon.$$ 

Taking $\epsilon = \frac{1}{2}$, the above implies that

$$2x_n \leq y_n,$$

for $n \geq N$. Theorem 3.6.4 implies that $\lim y_n = +\infty$, since in this part, we have $\lim x_n = +\infty$.

7b. Let $\epsilon > 0$. Suppose $|y_n| < M$ for $M \in \mathbb{R}$. There exists $N$ such that $n \geq N$ implies

$$0 \leq \frac{x_n}{y_n} \leq \frac{\epsilon}{M}.$$ 

Therefore $0 \leq x_n \leq \epsilon$, for $n \geq N$. This completes the proof.

SECTION 3.7

5. We claim that if $\sum x_n$ is convergent and $\sum y_n$ is divergent, then $\sum x_n + y_n$ must be divergent, so that no such example exists.

By way of contradiction, suppose $\sum x_n$ is convergent, $\sum y_n$ is divergent, and $\sum x_n + y_n$ are convergent. Let $X_n$, $Y_n$, and $S_n$ be the sequences of partial sums $\sum_1^n x_i$, $\sum_1^n y_i$, $\sum_1^n x_i + y_i$, respectively. Then clearly

$$S_n = X_n + Y_n,$$

or,

$$S_n - X_n = Y_n.$$
Note that $S_n - X_n$ is a convergent sequence (since it is a sum of convergent sequences), whereas $Y_n$ is divergent. This supplies the contradiction.

11. If $\sum a_n$, with $a_n > 0$, is convergent, then $\sum a^2_n$ is in fact convergent as well. There are several ways to prove this. Here is the quickest: since $\sum a_n$ converges, $a_n \to 0$. Then $\frac{a^2_n}{a_n} = a_n \to 0$, hence by the limit comparison test (Theorem 3.7.8), $\sum a^2_n$ converges.