Comments. On # 14 in Section 3.4, some of you had the right idea, but did not develop your proof fully. You have to define the subsequence before you can use it. Many proofs simply began by using a subsequence with certain properties, without proving the existence of such a subsequence.

1. **Section 3.3**

(8) Let $I_n = [a_n, b_n]$. The nested interval condition then shows that $a_n$ (resp. $b_n$) is a monotone increasing (resp. decreasing) sequence. Hence $a_n$ and $b_n$ converge, say, $a_n \to a$ and $b_n \to b$. Since

$$a_n \leq b_n,$$

taking limits gives $a \leq b$, i.e.,

$$a_n \leq a \leq b \leq b_n.$$

Hence, $a \in [a_n, b_n] = I_n$ for all $n$. It follows that $a \in \bigcap I_n$, so that the latter intersection is nonempty.

(12) Hints only.

(a) Write

$$\left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right),$$

and apply the product rules for limits and the definition of $e$.

(b) Write

$$\left(1 + \frac{1}{n}\right)^{2n} = \left(\left(1 + \frac{1}{n}\right)^n\right)^2,$$

and apply the limit rule for exponents.

(c) Write

$$\left(1 + \frac{1}{n+1}\right)^n = \left(1 + \frac{1}{n+1}\right)^{n+1} \left(1 + \frac{1}{n+1}\right)^{-1},$$

and apply product rule for limits, the fraction rule for limits, and the definition of $e$. 
(d) Write
\[
\left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n
\]
\[
= \left(\frac{n}{n-1}\right)^{-n}
\]
\[
= \left(1 + \frac{1}{n-1}\right)^{-n}.
\]

Apply similar techniques (factoring, exponents, limit theorems) as before.

2. Section 3.3

(6) Note that \(x_{n+1} < x_n\) if and only if
\[
\frac{x_{n+1}}{x_n} < 1,
\]
which holds if and only if
\[
\frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}} < 1};
\]
rearranging this yields the equivalent condition
\[
\left(n + 1\right)^n < n.
\]
This completes the proof of the equivalence of the two conditions.

Using that fact that the sequence on the LHS is a monotonically increasing sequence that converges to \(e\), it is obvious that the LHS is strictly less than 3. Hence \(x_{n+1} < x_n\) for \(n \geq 3\). Obviously \(x_n \geq 1\), so the monotone convergence theorem implies that \(x_n\) is convergent.

To compute the limit of the sequence, note that \(x_{2n}\) and \(x_n\) converge to this same limit as \(n \to \infty\). But
\[
x_{2n} = 2n^{\frac{1}{2n}} = 2^{\frac{1}{2n}} = 2^{\frac{1}{2n}} n^{\frac{1}{2n}} = 2^{\frac{1}{2}} \sqrt{x_n}.
\]

Taking limits (this is where we need to know that the sequence converges in the first place), we obtain \(L = \sqrt{L}\). Hence \(L = 0\) or \(L = 1\); since \(x_n \geq 1\) for all \(n\), we obtain \(L = 1\).

(14) Since \(s \not\in \{x_n\}\), we have \(x_n < s\) for all \(n\). This holds since \(x_n \leq s\) for all \(n\), by the definition of upper bound, and since \(x_n \neq s\) for any \(s\).
We claim that for any positive number \( k \), there is an \( x_{n_k} \) such that \( x_{n_k} \in (s - \frac{1}{k}, s) \), i.e., \( s - \frac{1}{k} < x_{n_k} < s \). We can prove this lemma by contradiction. If, for some \( k \), the interval \((s - \frac{1}{k}, s)\) contains no \( x_n \), then since \( x_n \leq s \) for all \( n \), we have \( x_n \leq s - \frac{1}{k} \) for all \( n \). Hence \( s - \frac{1}{k} \) is an upper bound for \( \{x_n\} \). Since \( \frac{1}{k} > 0 \), this contradicts the definition of supremum. This completes the proof.

Define \( x_{n_k} \) according to the above lemma. Since \( s - \frac{1}{k} < x_{n_k} < s \), the squeeze theorem implies that \( x_{n_k} \to s \).