Graded problems were Section 3.1 5a, 5d, 8, and 10. Some additional problems are included here.

1. Section 2.1

For parts of number (5), some of you tried a squeeze theorem argument. This is an example of limit theorem, not the definition of convergence, and is part of the next chapter. For these problems, you needed to verify the \( \epsilon - K \) definition directly (and that was just as easy).

(5a) Note that
\[
\frac{n}{n^2 + 1} \leq \frac{n}{n^2} = \frac{1}{n},
\]
so that
\[
\left| \frac{n}{n^2 + 1} - 0 \right| \leq \frac{1}{n}.
\]
Let \( \epsilon > 0 \). Then if \( n > \frac{1}{\epsilon} \), the above inequality shows that
\[
\left| \frac{n}{n^2 + 1} \right| \leq \epsilon.
\]
Hence \( \frac{n}{n^2 + 1} \to 0 \) as \( n \to \infty \).

(5b) Note that
\[
\left| \frac{2n}{n+1} - 2 = \frac{2n}{n+1} - \frac{2n + 2}{n+1} \right| = \left| \frac{-2}{n+1} \right| \leq \frac{2}{n}.
\]
Let \( \epsilon > 0 \). Then for \( n \geq \frac{2}{\epsilon} \), by the above,
\[
\left| \frac{2n}{n+1} - 2 \right| \leq \epsilon.
\]
This completes the proof.

(5c) Note that
\[
\left| \frac{3n + 1}{2n + 5} - \frac{3}{2} = \frac{6n + 2 - 6n - 15}{4n + 10} \right| = \frac{13}{2n + 5}.
\]
Hence if \( \epsilon > 0 \),
\[
\left| \frac{3n + 1}{2n + 5} - \frac{3}{2} \right| \leq \epsilon
\]
for all \( n > \frac{13}{2\epsilon} - \frac{5}{2} \).

(5d) Note that
\[
\left| \frac{n^2 - 1}{2n^2 + 3} - \frac{1}{2} \right| = \left| \frac{2n^2 - 2 - 2n^2 - 3}{2(2n^2 + 3)} \right| = \frac{5}{2(2n^2 + 3)} \leq \frac{5}{4n^2}.
\]

Let \( \epsilon > 0 \). Then for all \( n \geq \sqrt{\frac{5}{4\epsilon}} \),
\[
\left| \frac{n^2 - 1}{2n^2 + 3} - \frac{1}{2} \right| \leq \epsilon.
\]

(8) Lets look at the definitions of the two statements: We have
\[
\lim |x_n| = 0 \text{ if and only if }
\]
For all \( \epsilon > 0 \), there exists \( K \) such that \( n \geq K \) implies
\[
||x_n| - 0| \leq \epsilon,
\]
and \( \lim x_n = 0 \) if and only if
\[
\text{For all } \epsilon > 0, \text{ there exists } K \text{ such that } n \geq K \text{ implies } |x_n - 0| \leq \epsilon,
\]
Of course, \( |x_n - 0| = ||x_n| - 0| = |x_n| \), so these two statements are identical.

(10) Put \( \epsilon = \frac{x}{2} \). Since \( x_n \to x \), there is an \( M \) such that \( n \geq M \) implies
\[
|x_n - x| \leq \epsilon = \frac{x}{2}.
\]
In particular, this implies \( x - x_n \leq \frac{x}{2} \), i.e.,
\[
0 < \frac{x}{2} < x_n,
\]
for all \( n \geq M \).

Problem 18 is solved in a similar manner: exploit \( |x_n - x| \leq \frac{x}{2} \)
for \( n \geq M \) to obtain \( x - x_n \leq \frac{x}{2} \) and \( x_n - x \leq \frac{x}{2} \). The first inequality gives \( \frac{x}{2} \leq x_n \), and the second \( \frac{x}{2} \leq \frac{3}{2}x \).

(12) Note that
\[
|\sqrt{n^2 + 1} - n| = \left| (\sqrt{n^2 + 1} - n) \frac{\sqrt{n^2 + 1} + n}{\sqrt{n^2 + 1} + n} \right| = \frac{1}{\sqrt{n^2 + 1} + n} \leq \frac{1}{n}.
\]

Given \( \epsilon > 0 \), the above bound is \( \leq \epsilon \) for all \( n > \frac{1}{\epsilon} \).