Graded problems for this assignment were problems 7 and 20 of section 1.2, and problems 4, 9, and 20 on Section 2.1.

**General comments on writing proofs:** Your proofs should be readable. They should not look like scratch work. They should read like a proof in a math textbook. That is the point of this course. Try to refer when appropriate to any theorems you’ve proven in class or your textbook; if the result does not have an easy name (like “distributive property”), refer to the number. It is not necessary to do this 100% of the time, but if you complete your homework and have not done it once, you need to look back. This is especially important for simpler, more fundamental problems, like problem 20 in section 2.1. A precalculus student can solve that problem; the point is for you to prove that the statement follows in a logical way from the axioms or theorems in your coursework.

Your sentences should be complete, and you should be very careful with using logical symbols in place of words. Many of the proofs that I saw, which had no clear place to start reading or no clear direction to continue reading, received zero points fairly quickly.

Do not use three dots in a triangle to mean “imply”; the only logical symbols (i.e., symbols replacing words) you use should be \(\exists, \forall, \Rightarrow, \Leftarrow, \) and \(\Leftrightarrow\). But the best thing you can do is to use no logical symbols in your writeup - use words whenever you can. Maybe if you lose the instinct to throw symbols at the paper, you will start thinking better about what you need to do.

1. **Section 1.2, pp. 15**

*Note:* Both of these problems use induction. Any nonsensical hand-waving argument, along the lines of “I’ve checked the statement is true for \(n = 1, 2, ..., 10\), so by induction (sic) the proof is complete” received no credit. Another comment I have on your use of induction is that it is not necessary to use the set “\(S\)”. It is perfectly fine to write induction proofs as I do below. Using the set \(S\) without defining to be the set of all natural numbers for which the statement is true gives the
impression that you are just mimicking what is in the book, without necessarily understanding what it actually means.

(7) We proceed by induction. Clearly, $5^2 - 1 = 24$ is divisible by 8. This establishes the result for $n = 1$. Next, suppose the statement is true for $n = k$ (the “induction hypothesis”). We will show that this implies that the statement is true for $n = k + 1$, completing the inductive proof.

There are several ways to do this. One way is to “add and subtract”:

$$5^{2k + 2} - 1 = 5^2 \times 5^{2k} - 1 = 5^2 \times 5^{2k} - 5^2 + 5^2 - 1 = 5^2(5^{2k} - 1) + 24.$$ 

Since the first term is divisible by 8 (this is where the induction hypothesis is used) and so is 24, we are done. Another way to do this would be to write (using the induction hypothesis) $5^{2k} - 1 = 8m$ for $m \in \mathbb{N}$. Then rearrange to get $5^{2k} = 8m - 1$, substitute this above, and obtain

$$5^{2k+2} - 1 = 25(8m - 1) - 1 = 25 \times 8m - 24.$$ 

Some of you seemed to be doing something similar, but you used very large numbers for some reason. Another approach some of you used was to factor $5^{2(k+1)} - 1$ as $(5^{k+1} - 1)(5^{k+1} + 1)$. I did not give credit for this because what followed was a long, meandering discussion that rested on an unproven assertion, namely, “out of two consecutive even integers, one of them is divisible by 4”.

(20) This can also be solved using induction. Note that since the pattern does not start until $n = 3$, we need to verify by hand that the statement $n = 1$ and $n = 2$ are true, but these cases are trivial. (You should still write them).

Suppose the result is true for $k \leq n$, where $n \leq 2$. We will prove that this implies the result is true for $k = n + 1$. The induction hypothesis implies, in particular (taking $k = n, n - 1$), that

$$1 \leq x_n \leq 2,$$
$$1 \leq x_{n-1} \leq 2.$$ 

Adding these gives (at this point, you could cite Theorem 2.1.7 (b))

$$2 \leq x_n + x_{n-1} \leq 4,$$

and dividing by 2 and applying Theorem 2.1.7 (c), we get

$$1 \leq x_{n+1} \leq 2.$$ 

This completes the inductive proof.
I did not give any credit for the statement, “if two numbers are between 1 and 2, so is their average”. This is not a proof – it is a repetition of what you are supposed to show. You’ve just repeated the problem, using words.

2. Section 2.1, pp. 30

(4) You cannot use induction on $a$ here unless you first prove that $a$ is an integer!

This is an example of a basic problem that is meant to test your knowledge of the axioms.

By (2.1.1, A4), $a^2 = a$ implies $a^2 - a = 0$. By the distributive property, this implies $a(a - 1) = 0$.

Now you need to include a quick proof that “$ab = 0$ implies $a = 0$ or $b = 0$”. This is by cases; if $a \neq 0$, then multiplying by $a^{-1}$ (which exists by (M4)), we get $b = 0$; if $b \neq 0$, multiply by $b^{-1}$ to get $a = 0$.

Therefore, $a = 0$ or $a - 1 = 0$ ( $\implies$ $a = 1$), which is the result.

(9) Let $x_1, x_2 \in K$. Then for $s_1, s_2, t_1, t_2 \in Q$, we have

$$x_1 = s_1 + t_1 \sqrt{2}, x_2 = s_2 + t_2 \sqrt{2}.$$  

The task is to prove that $x_1 + x_2, x_1 x_2 \in K$. We will repeatedly use the Axioms in 2.1.1.

We have

$$x_1 + x_2 = (s_1 + s_2) + (t_1 + t_2) \sqrt{2},$$

and

$$x_1 x_2 = (s_1 + t_1 \sqrt{2})(s_2 + t_2 \sqrt{2}) = s_1 s_2 + 2 t_1 t_2 + (t_1 s_2 + t_2 s_1) \sqrt{2},$$

hence both are in $K$, since $Q$ is closed under addition and multiplication (this is extremely easy to prove).

Part (b) asks you to assume $x = s + t \sqrt{2} \neq 0$ and to prove $x^{-1} \in K$. Of course

$$\frac{1}{x} = \frac{1}{s + t \sqrt{2}}.$$
which is not clearly of the form needed. Multiplying both sides by the conjugate \( s - t\sqrt{2} \), we get

\[
\frac{1}{x} = \frac{s - t\sqrt{2}}{s^2 - 2t^2} = \frac{s}{s^2 - 2t^2} - \frac{t}{s^2 - 2t^2}\sqrt{2},
\]

provided \( s^2 - 2t^2 \neq 0 \). If we can prove that, we are done.

We can do this reductio ad absurdum. Suppose that it is zero. If one of \( s, t \) is zero, then clearly, so is the other one. Since \( x \neq 0 \), this is impossible, so both of \( s, t \) are not zero. Then we may write \( s^2/t^2 = 2 \), and since \( Q \) is closed under multiplication, this implies that \( \sqrt{2} \) is rational. This is a contradiction (you have specifically proven that \( \sqrt{2} \) is irrational in class, I hope), so \( s^2 - 2t^2 \) cannot be zero.

If you did not prove that the above denominator is not zero, I think I gave you half credit; that is the only part of this problem that is interesting.

(20) (a) It suffices to prove simply that \( c^2 < c \), since Theorem 2.1.8 and Theorem 2.1.7a will then give \( 0 < c^2 < c < 1 \). Of course, since \( 0 < c \), we can multiply \( c < 1 \) by \( c \) (using Theorem 2.1.7c ) to get \( c^2 < c \). We are done.

(b) Again, by Theorem 2.1.7 (a), we only need to prove \( c^2 < c \). By Theorem 2.1.7, we have \( 0 < c \) as a consequence of \( 1 < c \) and \( 0 < 1 \), so multiplying by \( 1 < c \) by \( c \) gives \( c < c^2 \).

Many of you, for some reason, decided to \( c = \frac{m}{n} \), and then work with \( m \) and \( n \) instead of \( c \). This does not help you in any way and puts two pieces on the board instead of one. In that case, your proofs were very difficult to follow and you recieved at most half credit.