Solution for Homework 11
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36.1-7) Let $L_1$ and $L_2$ be two languages in $P$.
   
   $L_1$ in $P$ $\Rightarrow$ There exists a polynomial-time algorithm $A$ that decides $L_1$.
   
   $L_2$ in $P$ $\Rightarrow$ There exists a polynomial-time algorithm $B$ that decides $L_2$.

**Closure under UNION** - The following polynomial-time algorithm decides $L_1$ UNION $L_2$:
   
   Let $w$ be an arbitrary instance (encoding) of the problem
   Run $A$ with input $w$
   If $A$ accepted $w$
       Then
           Accept
   Else
       Run $B$ with input $w$
       If $B$ accepted $w$
           Then
               Accept
       Else
           Reject

**Closure under INTERSECTION** - The following polynomial-time algorithm decides $L_1$ INTERSECT $L_2$:
   
   Let $w$ be an arbitrary instance (encoding) of the problem
   Run $A$ with input $w$
   Run $B$ with input $w$
   If ($A$ accepted $w$) AND ($B$ accepted $w$) Then
       Accept
   Else
       Reject

**Closure under CONCATENATION** - The following polynomial-time algorithm decides the $L_1 L_2$:
   
   Let $w$ be an arbitrary instance (encoding) of the problem
   For all $w'$, $w''$ such that ($w = w' w''$) and ($w'$ in $L_1$) and ($w''$ in $L_2$)
       Run $A$ with input $w'$
       Run $B$ with input $w''$
       If ($A$ accepted $w'$) AND ($B$ accepted $w''$) Then
           Accept & Exit
       Else
           Reject

**Closure under COMPLEMENT** - The following polynomial-time algorithm decides the COMPLEMENT of $L_1$:
   
   Let $w$ be an arbitrary instance (encoding) of the problem
   Run $A$ with input $w$
   If ($A$ accepted $w$) Then
       Accept
   Else
       Reject

**Closure under Kleen star** – Using closure under UNION and CONCATENATION we conclude that class $P$ is closed under Kleen star.
36.2-9) Let L be any language in P. Therefore:

\[ \neg L \subseteq P \quad (\text{where } \neg L \text{ is the complement of } P) \]
\[ \Rightarrow \neg L \subseteq \text{NP} \quad (\text{since } P \text{ is a subset of NP}) \]
\[ \Rightarrow L \in \text{co-NP} \]

Since L is any language in P we conclude that P is a subset of co-NP.

36.3-1)

\( L_1 \leq_p L_2 \Rightarrow \) There exists a polynomial-time computable function \( f : \{0, 1\}^* \rightarrow \{0, 1\}^* \) such that for all \( x \) in \( \{0, 1\}^* \), \( x \in L_1 \) if and only if \( f(x) \in L_2 \).

\( L_2 \leq_p L_3 \Rightarrow \) There exists a polynomial-time computable function \( g : \{0, 1\}^* \rightarrow \{0, 1\}^* \) such that for all \( y \) in \( \{0, 1\}^* \), \( y \in L_2 \) if and only if \( g(y) \in L_3 \).

Therefore, there exists a polynomial-time computable function \( h : \{0, 1\}^* \rightarrow \{0, 1\}^* \) such that for all \( x \) in \( \{0, 1\}^* \), \( x \in L_1 \) if and only if \( h(x) = g(f(x)) = g(y) \in L_3 \). In other words, to reduce \( L_1 \) to \( L_3 \) in polynomial time, first use \( f(x) \) to reduce \( L_1 \) to \( L_2 \) and then use \( g(y) \) (\( g \) with the input \( f(x) \)) to reduce \( L_2 \) to \( L_3 \).