**Probabilistic Independence**

Consider the following drawing:

![Diagram](image)

In this drawing, we see that $|E| = n + e$, $|F| = w + n$, $|E \cap F| = n$, and $|S| = n + s + e + w$.

Now, from Conditional Probability, we have that: $P(E \mid F) = P(E \cap F) / P(F)$, and when $E$ is independent of $F$, we have that: $P(E \mid F) = P(E)$, so combining these equations yields:

$$P(E) = P(E \mid F) = P(E \cap F) / P(F),$$

hence $P(E)P(F) = P(E \cap F)$.

**This is our test for Independence:** $P(E \cap F) = P(E)P(F)$.

**What This Really Means:**

If we replace these terms using their equivalent cardinality forms:

$$P(E \cap F) = |E \cap F| / |S|, \quad P(E) = |E| / |S|, \quad \text{and} \quad P(F) = |F| / |S|$$

we get:

$$P(E \cap F) = |E \cap F| / |S| = (|E| / |S|)(|F| / |S|) = P(E)P(F)$$

so multiplying by $|S|^2$ yields:

$$|E \cap F| \times |S| = |E| \times |F|.$$  

In terms of the $n$, $s$, $e$, and $w$ regions, this becomes:

$$n(n + s + e + w) = (n + e)(n + w)$$

so

$$nn + ns + ne + nw = nn + ne + nw + ew$$

hence

$$ns = ew, \text{ which is equivalent to } n / w = e / s.$$  

So, what is Independence of $E$ with respect to $F$? The ratio $(n / w)$ compares the part of $E$ within $F$ to the rest of $F$, and the ratio $(e / s)$ compares the part of $E$ outside $F$ to the rest of the things outside $F$.

**Conclusion:** $E$ is independent of $F$ when the ratio of $E$ in $F$ to the rest of $F$ equals the ratio of $E$ outside of $F$ to the rest of the things outside of $F$. That is, $E$ is equally likely within $F$ or outside $F$. 
