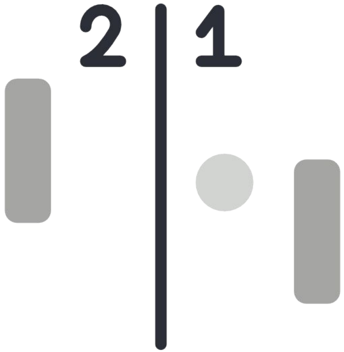
Chapter 3: Finite Markov Decision Processes

Seungjae Ryan Lee

endtoend.ai

Markov Decision Process (MDP)

- Simplified, flexible reinforcement learning problem
- Consists of States $\mathcal{S}$, Actions $\mathcal{A}$, Rewards $\mathcal{R}$

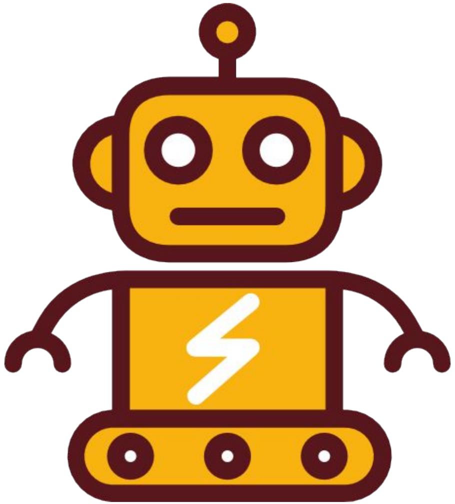


States
Info available to agent

Actions
Choice made by agent

Rewards
Basis for evaluating choices

Agent



The learner
Takes action

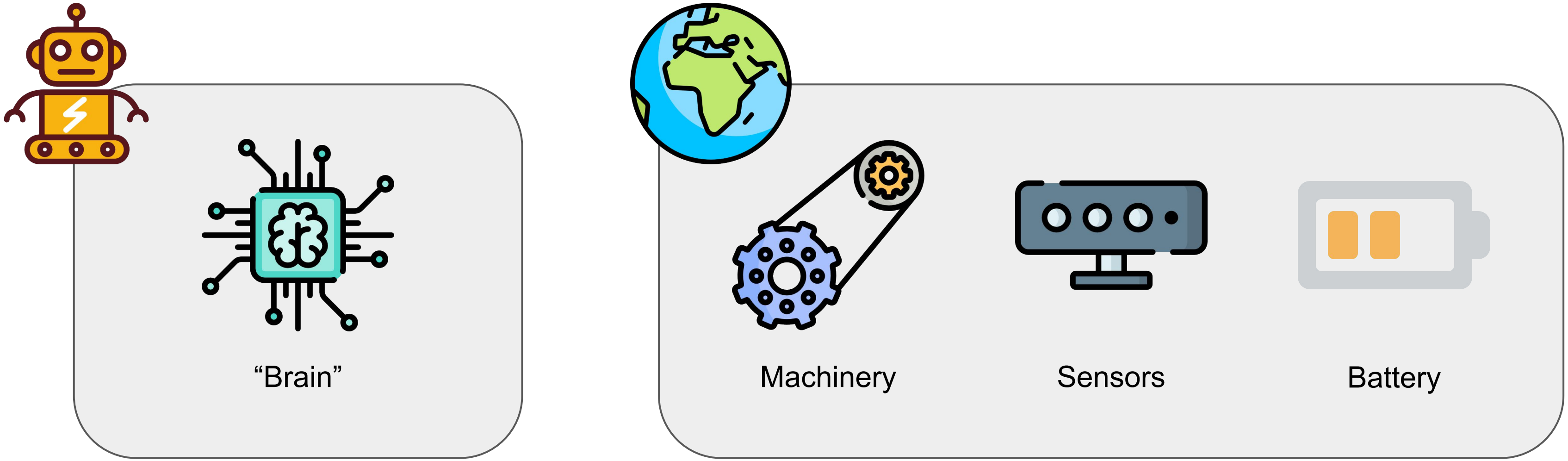
Environment

Everything outside the agent
Returns state and reward

endtoend.ai

Agent-Environment Boundary

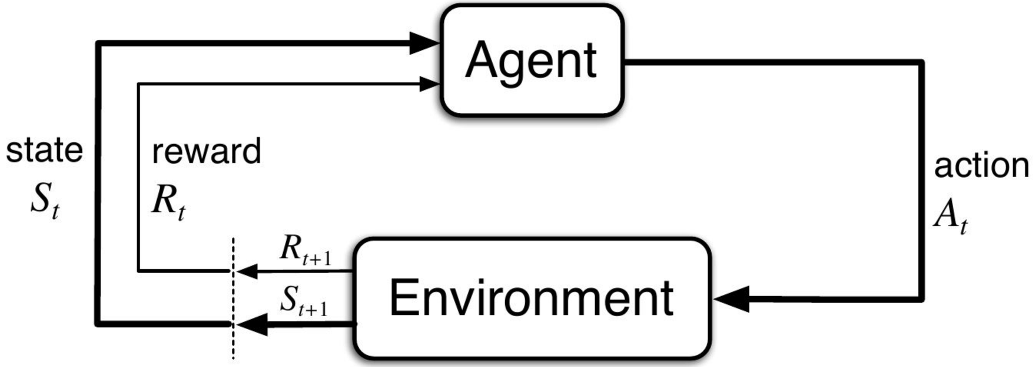
- Anything the agent cannot **arbitrarily change** is part of the environment
  - Agent might still **know** everything about the environment
- Different boundaries for different purposes



endtoend.ai

Agent-Environment Interactions

1. Agent observes a state S_0
2. Agent takes action A_0
3. Agent receives reward R_1 and new state S_1
4. Agent takes another action A_1
5. Repeat



endtoend.ai

Transition Probability

- Probability of reaching state s' and reward r by taking action a on state s
- **Fully describes the dynamics of a finite MDP**

$$p(s', r \mid s, a) := \Pr\{S_t = s', R_t = r \mid S_{t-1} = s, A_t = a\}$$

- Can deduce other properties of the environment

$$p(s' \mid s, a) := \Pr\{S_t = s' \mid S_{t-1} = s, A_t = a\} = \sum_{r \in \mathcal{R}} p(s', r \mid s, a)$$

Expected Rewards

- Expected reward of taking action a on state s

$$r(s,a) := \mathbb{E}[R_t \mid S_{t-1} = s, A_{t-1} = a] = \sum_{r \in \mathcal{R}} r \sum_{s' \in \mathcal{S}} p(s', r \mid s, a)$$

- Expected reward of arriving in state s' by taking action a on state s

$$r(s,a,s') := \mathbb{E}[R_t \mid S_{t-1} = s, A_{t-1} = a, S_t = s'] = \sum_{r \in \mathcal{R}} r \frac{p(s', r \mid s, a)}{p(s' \mid s, a)}$$

endtoend.ai

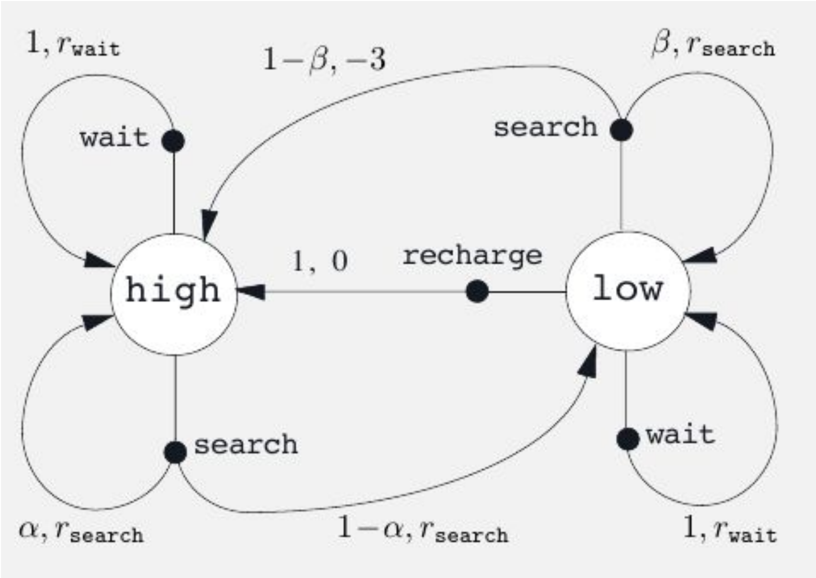
Recycling Robot Example

- States: Battery status (high or low)
- Actions
  - Search: High reward. Battery status can be lowered or depleted.
  - Wait: Low reward. Battery status does not change.
  - Recharge: No reward. Battery status changed to high.
- If battery is depleted, -3 reward and battery status changed to high.

s	a	s'	$p(s' \mid s, a)$	$r(s, a, s')$
high	search	high	α	$r_{\texttt{search}}$
high	search	low	$1 - \alpha$	$r_{\texttt{search}}$
low	search	high	$1 - \beta$	-3
low	search	low	β	$r_{\texttt{search}}$
high	wait	high	1	$r_{\texttt{wait}}$
high	wait	low	0	$r_{\texttt{wait}}$
low	wait	high	0	$r_{\texttt{wait}}$
low	wait	low	1	$r_{\texttt{wait}}$
low	recharge	high	1	0
low	recharge	low	0	0

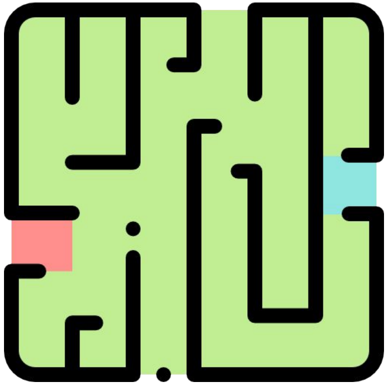
Transition Graph

- Graphical summary of MDP dynamics

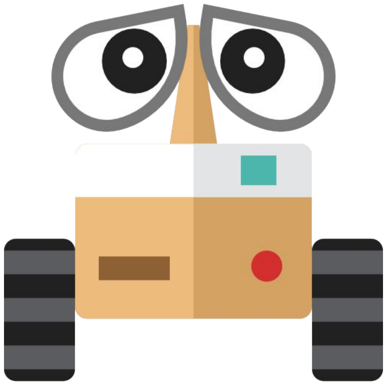


Designing Rewards

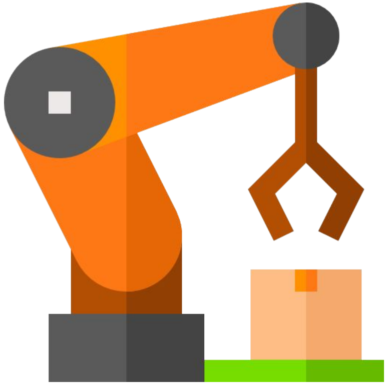
- Reward hypothesis
  - Goals and purposes can be represented by maximization of cumulative reward
- Tell *what* you want to achieve, not *how*



Always -1



Proportional to
forward action

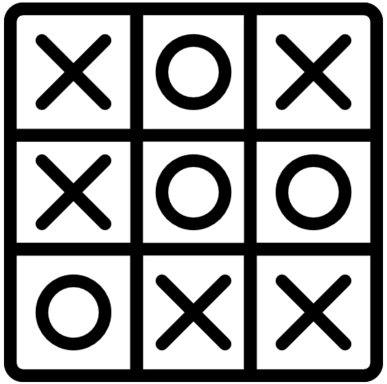


+1 for each box

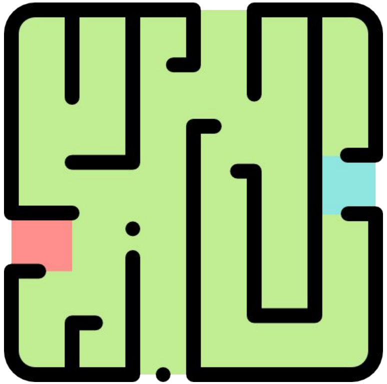
endtoend.ai

Episodic Tasks

- Interactions can be broken into *episodes*
- Episodes end in a special *terminal state*
- Each episode is independent



Finished when the game ends



Finished when the agent is out of the maze

Return for Episodic Tasks

- Sum of rewards from time step t
- Time of termination: T

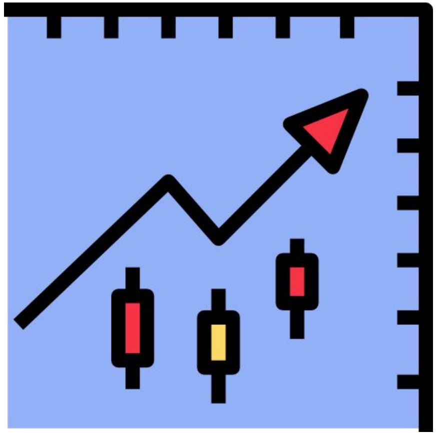
$$G_t = R_{t+1} + R_{t+2} + \ldots R_T$$

$$G_t = \sum_{k=t+1}^{T} R_k$$

endtoend.ai

Continuing Tasks

- Cannot be naturally broken into episodes
- Goes on without limit



Stock Trading

Return for Continuing Tasks

- Sum of rewards is almost always infinite
- Need to *discount* future rewards by factor $0 \leq \gamma < 1$
    - If $\gamma = 0$, the return only considers immediate reward (*myopic*)

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} \dots$$

$$G_t = \sum_{k=t+1}^{\infty} \gamma^{k-t-1} R_k$$

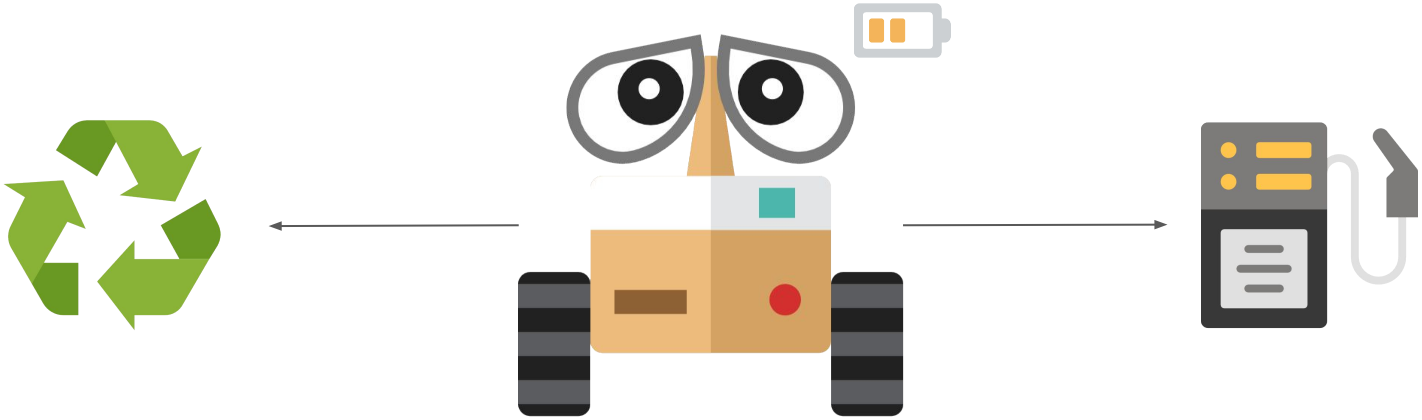
Unified Notation for Return

- Cumulative reward
- T can be a finite number or infinity
- Future rewards can be *discounted* with factor γ
  - If $T = \infty$, then γ must be less than 1.

$$G_t := \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$$

Policy

- Mapping from states to probabilities of selecting each possible action
- $\pi(a \mid s)$: Probability of selecting action a in state s



State-value function

- Expected return from state s and following policy π

$$v_\pi := \mathbb{E}_\pi[G_t \mid S_t = s]$$
$$:= \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s\right]$$

Action-value function

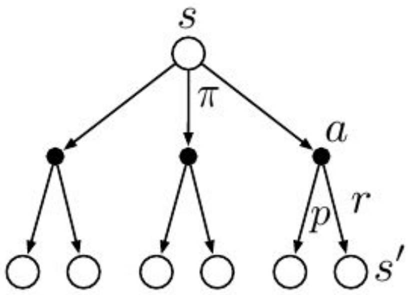
- Expected return from taking action a in state s and following policy π

$$q_\pi := \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a]$$
$$:= \mathbb{E}_\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s, A_t = a \right]$$

endtoend.ai

Bellman Equation

- Recursive relationship between $v_\pi(s)$ and $v_\pi(s')$



$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma v_\pi(s') \right]$$

Optimal Policies π_* and Value Functions v_*, q_*

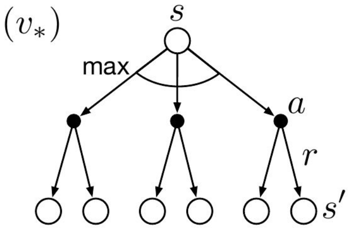
- For any policy π, $v_{\pi_*}(s) \geq v_\pi(s)$ for all states s
- There can be multiple optimal policies
- All optimal policies share same optimal value functions:

$$v_*(s) \doteq \max_\pi v_\pi(s)$$

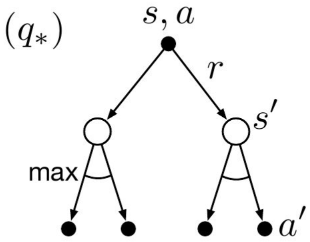
$$q_*(s, a) \doteq \max_\pi q_\pi(s, a)$$

Bellman Optimality Equation

- Bellman Equation for optimal policies



$$v_*(s) = \max_a \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma v_*(s')\right]$$



$$q_*(s, a) = \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma \max_{a'} q_*(s', a')\right]$$

endtoend.ai

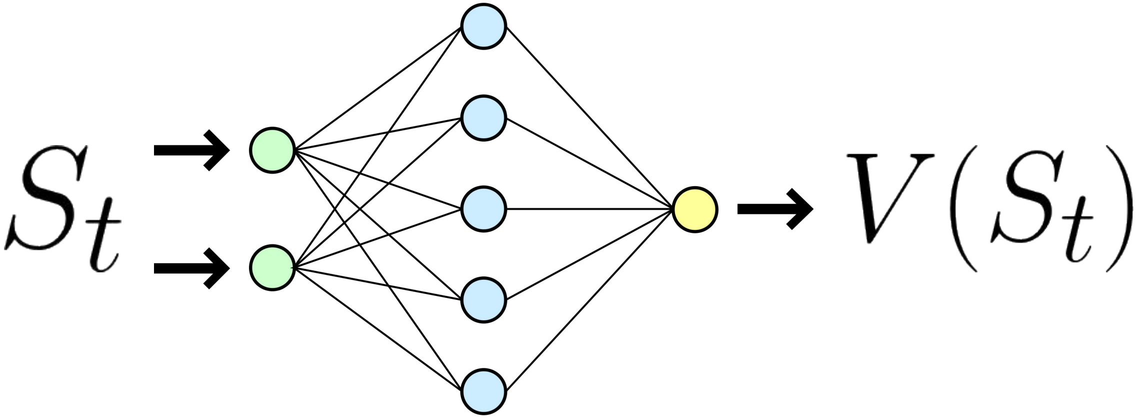
Solving Bellman Optimality Equation

- Linear system: $|\mathcal{S}|$ equations, $|\mathcal{S}|$ unknowns
- Possible to find the exact optimal policy
- Impractical in most environments
  - Need to know the dynamics of the environment
  - Need extreme computational power
  - Need Markov property

→ In most cases, approximation is the best possible solution.

Approximation

- Does not require complete knowledge of environment
- Less memory and computational power needed
- Can focus learning on frequently encountered states



Thank you!

Original content from

- [Reinforcement Learning: An Introduction by Sutton and Barto]

You can find more content in

- github.com/seungjaeryanlee
- www.endtoend.ai