Probabilistic Graphical Models

CMSC 478

UMBC

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A graph G that represents a probability distribution over random variables $X_1, \dots, X_N$

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Vertices $\leftrightarrow$ random variables

Edges show dependencies among random variables

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Two main flavors: *directed* graphical models and *undirected* graphical models

Outline

Directed Graphical Models

- Naïve Bayes

Undirected Graphical Models

- Factor Graphs
- Ising Model

Message Passing: Graphical Model Inference

Directed Graphical Models

A *directed* (acyclic) graph G=(V,E) that represents a probability distribution over random variables $X_1, \ldots, X_N$

Joint probability factorizes into factors of X_i conditioned on the parents of X_i

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Benefit: read the independence properties are *transparent*

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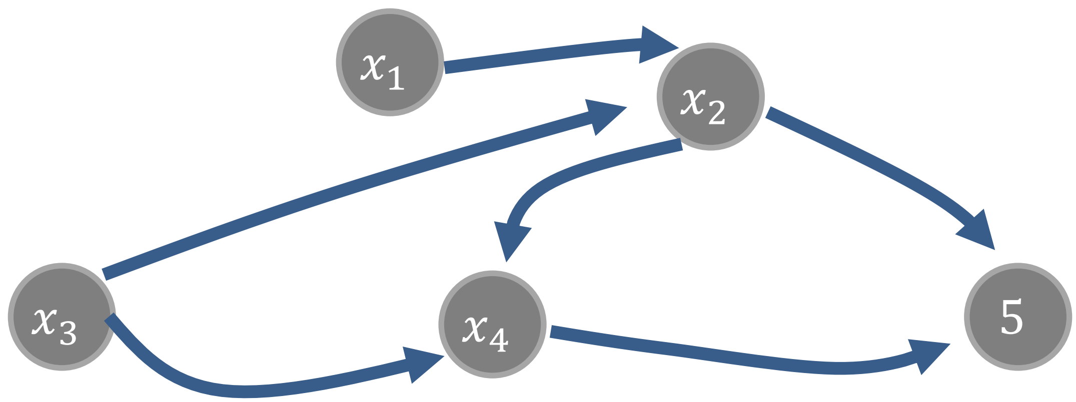
Joint probability factorizes into factors of X_i conditioned on the parents of X_i

A graph/joint distribution that follows this is a **Bayesian network**

Bayesian Networks: Directed Acyclic Graphs

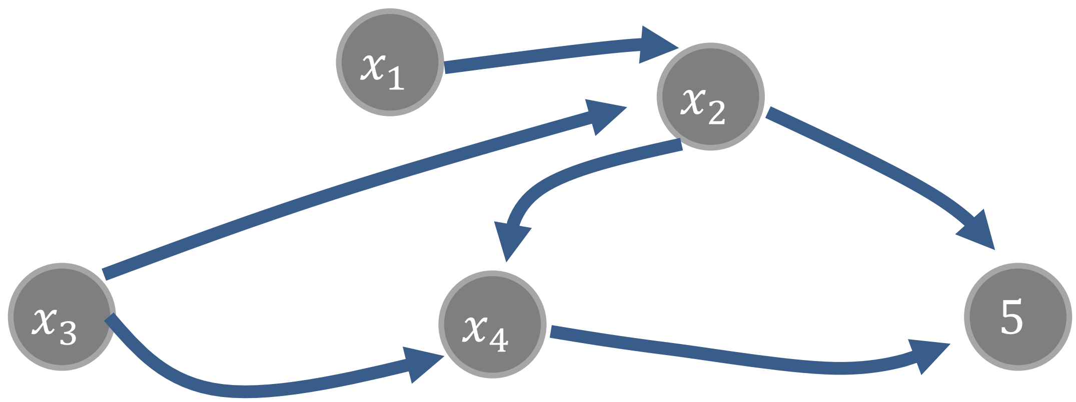


$$p(x_1, x_2, x_3, \ldots, x_N) = \prod_i p(x_i \mid \pi(x_i))$$

"parents of"

topological
sort

Bayesian Networks: Directed Acyclic Graphs



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$p(x_1, x_2, x_3, x_4, x_5)$= ???

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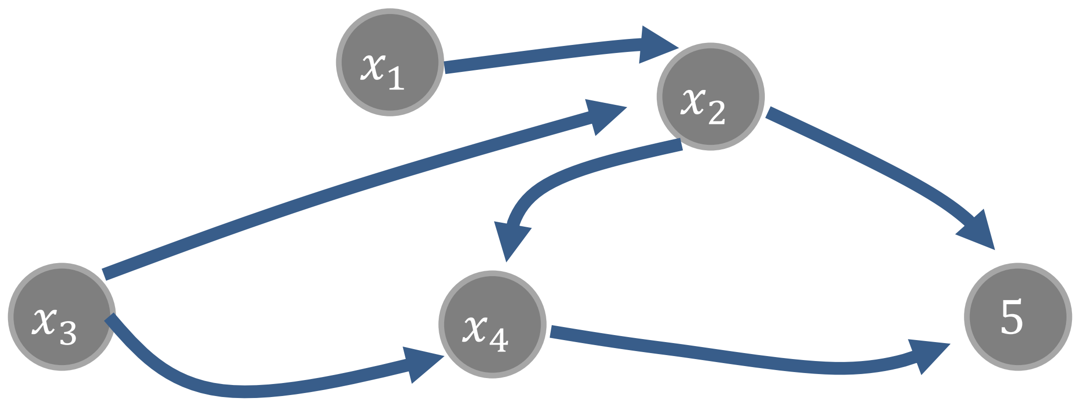
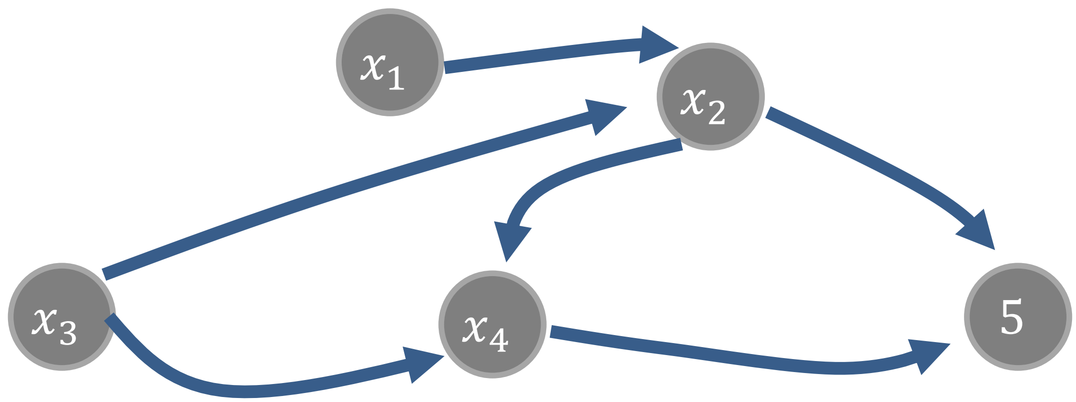


$$p(x_1, x_2, x_3, x_4, x_5) =$$
$$p(x_1)p(x_3)p(x_2|x_1, x_3)p(x_4|x_2, x_3)p(x_5|x_2, x_4)$$

Bayesian Networks: Directed Acyclic Graphs



$$p(x_1, x_2, x_3, \dots, x_N) = \prod_i p(x_i \mid \pi(x_i))$$

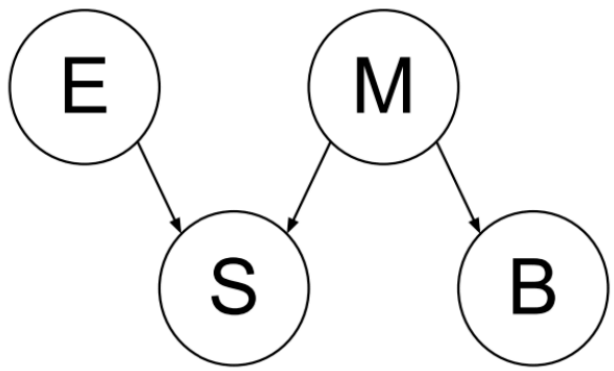
exact inference in general DAGs is NP-hard

inference in trees can be exact

A smell of sulphur (S) can be caused either by rotten eggs (E) or as a sign of the doom brought by the Mayan Apocalypse (M). The Mayan Apocalypse also causes the oceans to boil (B). The Bayesian network and corresponding conditional probability tables for this situation are shown below. The notation $+x$ means that x is true, and $-x$ means that x is false. For each part, you should give either a numerical answer (e.g., 0.81) or an arithmetic expression in terms of numbers from the tables below (e.g., 0.9 * 0.9). Note that the latter is easier and perfectly OK.

$P(E)$	
$+e$	0.4
$-e$	0.6

$P(S\|E,M)$			
$+e$	$+m$	$+s$	1.0
$+e$	$+m$	$-s$	0.0
$+e$	$-m$	$+s$	0.8
$+e$	$-m$	$-s$	0.2
$-e$	$+m$	$+s$	0.3
$-e$	$+m$	$-s$	0.7
$-e$	$-m$	$+s$	0.1
$-e$	$-m$	$-s$	0.9



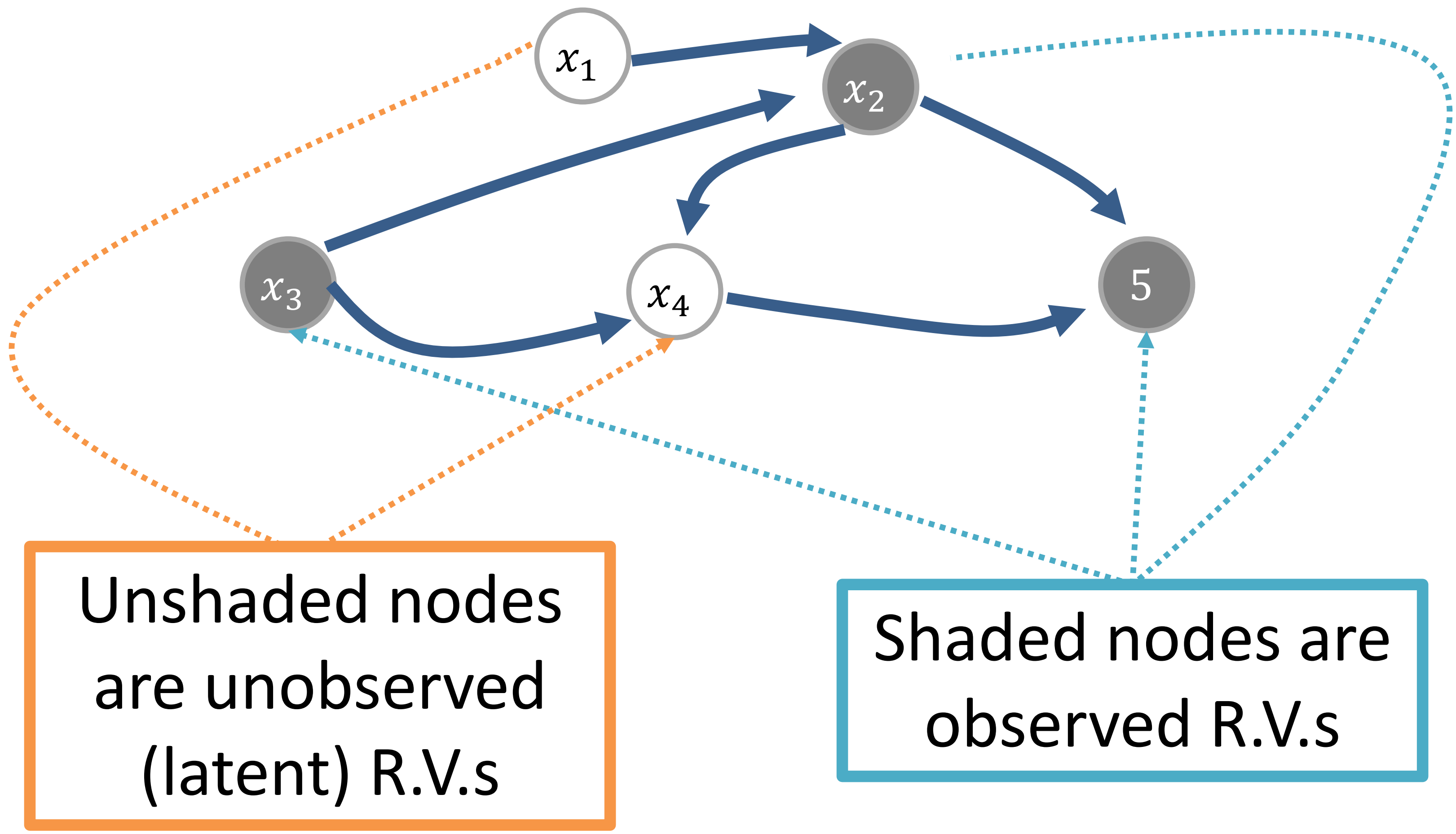
$P(M)$	
$+m$	0.1
$-m$	0.9

$P(B\|M)$		
$+m$	$+b$	1.0
$+m$	$-b$	0.0
$-m$	$+b$	0.1
$-m$	$-b$	0.9

(A) (X points) Compute the following entry from the joint distribution: $P(-e, -s, -m, -b)$

(B) (X points) What is the probability that the oceans boil?

Directed Graphical Model Notation



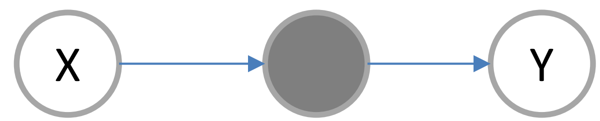
D-Separation: Testing for Conditional Independence

Variables X & Y are conditionally independent given Z if all (undirected) paths from (any variable in) X to (any variable in) Y are d-separated by Z

d-separation

X & Y are d-separated if for **all** paths P, one of the following is true:

P has a chain with an observed middle node



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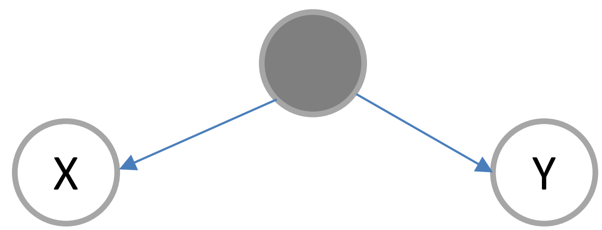
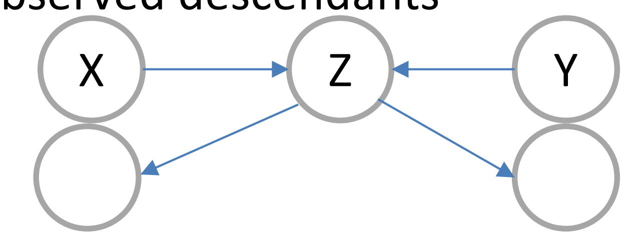


P includes a "v-structure" or "collider" with all unobserved descendants



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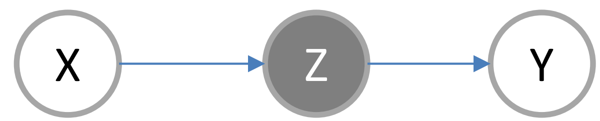
observing Z blocks the path from X to Y



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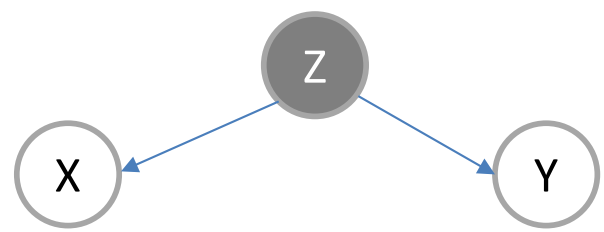

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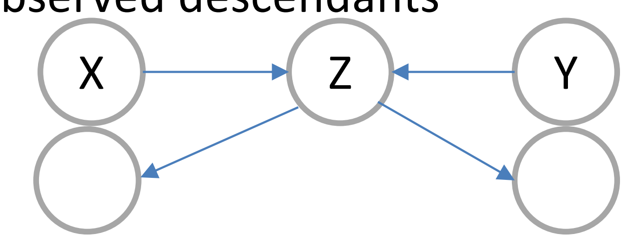
not observing Z blocks the path from X to Y



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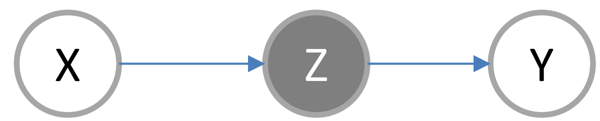

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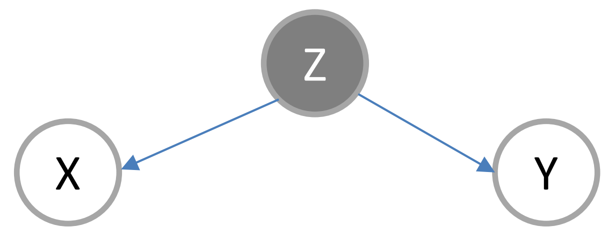
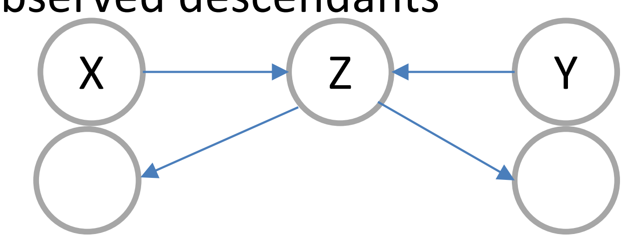


P includes a "v-structure" or "collider" with all unobserved descendants

not observing Z blocks the path from X to Y

$$p(x,y,z) = p(x)p(y)p(z|x,y)$$

$$p(x,y) = \sum_z p(x)p(y)p(z|x,y) = p(x)p(y)$$



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Naïve Bayes

$$\mathrm{argmax}_Y\, p(Y \mid X)$$

Apply Bayes rule and take logs

$$\mathrm{argmax}_Y \log p(X \mid Y) + \log p(Y)$$

likelihood *prior*

Naïve Bayes

$$\text{argmax}_Y\, p(Y \mid X)$$

Apply Bayes rule and take logs

$$\text{argmax}_Y \log p(X \mid Y) + \log p(Y)$$

Represent *X* is a D-dimensional vector (of features):

$$X = (X_1, X_2, X_3, \ldots, X_D)$$

Naïve Bayes

$$\text{argmax}_Y\, p(Y \mid X)$$

Apply Bayes rule and take logs

$$\text{argmax}_Y \log p(X \mid Y) + \log p(Y)$$

Naively generate each "feature" of *X*, conditioned on Y

$$\text{argmax}_Y \sum_{j=1}^{D} \log p(X_j|Y) + \log p(Y)$$

The Bag of Words Representation

I love this movie! It's sweet, but with satirical humor. The dialogue is great and the adventure scenes are fun... It manages to be whimsical and romantic while laughing at the conventions of the fairy tale genre. I would recommend it to just about anyone. I've seen it several times, and I'm always happy to see it again whenever I have a friend who hasn't seen it yet!

Adapted from Jurafsky & Martin (draft)

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it	6
I	5
the	4
to	3
and	3
seen	2
yet	1
would	1
whimsical	1
times	1
sweet	1
satirical	1
adventure	1
genre	1
fairy	1
humor	1
have	1
great	1
...	...

Bag of Words Representation

classifier

$\gamma($

seen	2
sweet	1
whimsical	1
recommend	1
happy	1
...	...

$) = c$

classifier

Adapted from Jurafsky & Martin (draft)

Naïve Bayes: A Generative Story

Generative Story

$\phi =$ distribution over K labels

for label $k = 1$ to K:

  $\theta_k =$ generate parameters

$p(y = k)$

$p(x_{ij} | y = k)$

global parameters

$$\sum_{j=1}^{D} \log p(X_{ij} | Y_i) + \log p(Y_i)$$

Naïve Bayes: A Generative Story

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$\phi =$ distribution over K labels

for label $k = 1$ to K:

$\quad \theta_k =$ generate parameters

for item $i = 1$ to N:

$\quad y_i \sim \text{Cat}(\phi)$

Choose the label

$$\sum_{j=1}^{D} \log p(X_{ij}|Y_i) + \log p(Y_i)$$

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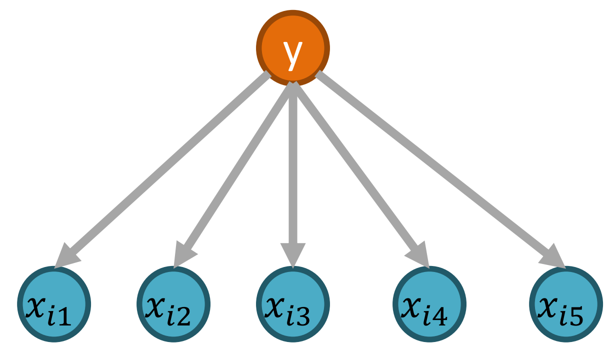
local variables

for each feature j

$x_{ij} \sim \text{F}_j(\theta_{y_i})$



Generate each feature based on the label

$$\sum_{j=1}^{D} \log p(X_{ij} | Y_i) + \log p(Y_i)$$

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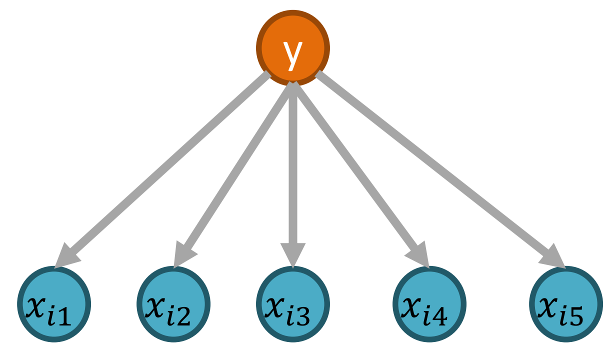

each x_{ij} is conditionally independent of one another (given the label)

$$\sum_{j=1}^{D} \log p(X_{ij}|Y_i) + \log p(Y_i)$$

Naïve Bayes: A Generative Story

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for label $k = 1$ to K:

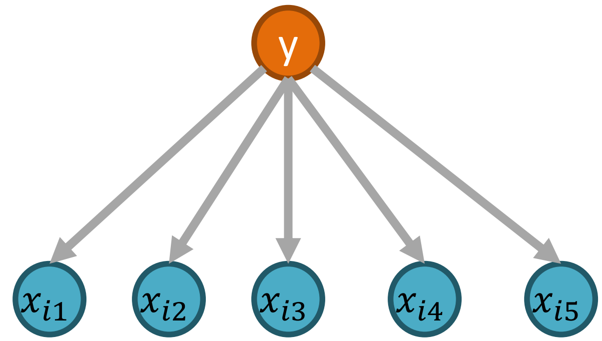
$\quad\theta_k =$ generate parameters

for item $i = 1$ to N:

$\quad y_i \sim \mathrm{Cat}(\phi)$

$\quad$for each feature j

$\quad\quad x_{ij} \sim \mathrm{F}_j(\theta_{y_i})$



Maximize Log-likelihood

$$\mathcal{L}(\theta) = \sum_i \sum_j \log F_{y_i}(x_{ij}; \theta_{y_i}) + \sum_i \log \phi_{y_i} \text{ s.t.}$$

$$\sum_k \phi_k = 1 \qquad \phi_k \geq 0 \qquad \theta_k \text{ is valid for } F_j$$

Multinomial Naïve Bayes: A Generative Story

Generative Story

ϕ = distribution over K labels

for label $k = 1$ to K:

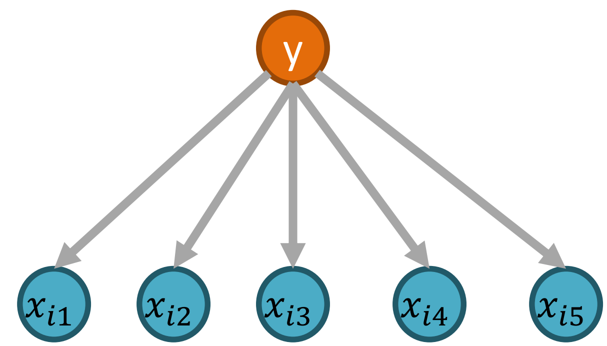
θ_k = distribution over J feature values

for item $i = 1$ to N:

$y_i \sim \text{Cat}(\phi)$

for each feature j

$x_{ij} \sim \text{Cat}(\theta_{y_i})$



Maximize Log-likelihood

$$\mathcal{L}(\theta) = \sum_i \sum_j \log \theta_{y_i, x_{i,j}} + \sum_i \log \phi_{y_i} \quad \text{s.t.}$$

$$\sum_k \phi_k = 1 \qquad \phi_k \geq 0 \qquad \sum_j \theta_{kj} = 1 \; \forall k \qquad \theta_{kj} \geq 0,$$

Multinomial Naïve Bayes: A Generative Story

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for item $i = 1$ to N:

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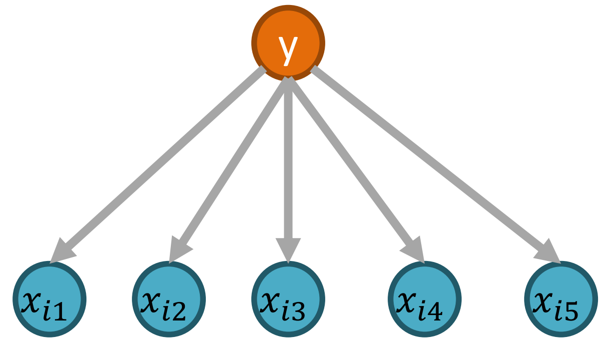
for each feature j

$x_{ij} \sim \text{Cat}(\theta_{y_i, j})$



Maximize Log-likelihood via Lagrange Multipliers ($\geq$ 0 constraints not shown)

$$\mathcal{L}(\theta) = \sum_i \sum_j \log \theta_{y_i, x_{i,j}} + \sum_i \log \phi_{y_i} - \mu\left(\sum_k \phi_k - 1\right) - \sum_k \lambda_k \left(\sum_j \theta_{kj} - 1\right)$$

Multinomial Naïve Bayes: Learning

Calculate *class priors*

For each k:

$items_k$ = all items with class = k

$$p(k) = \frac{|\text{items}_k|}{\#\ \text{items}}$$

Calculate feature generation terms

For each k:

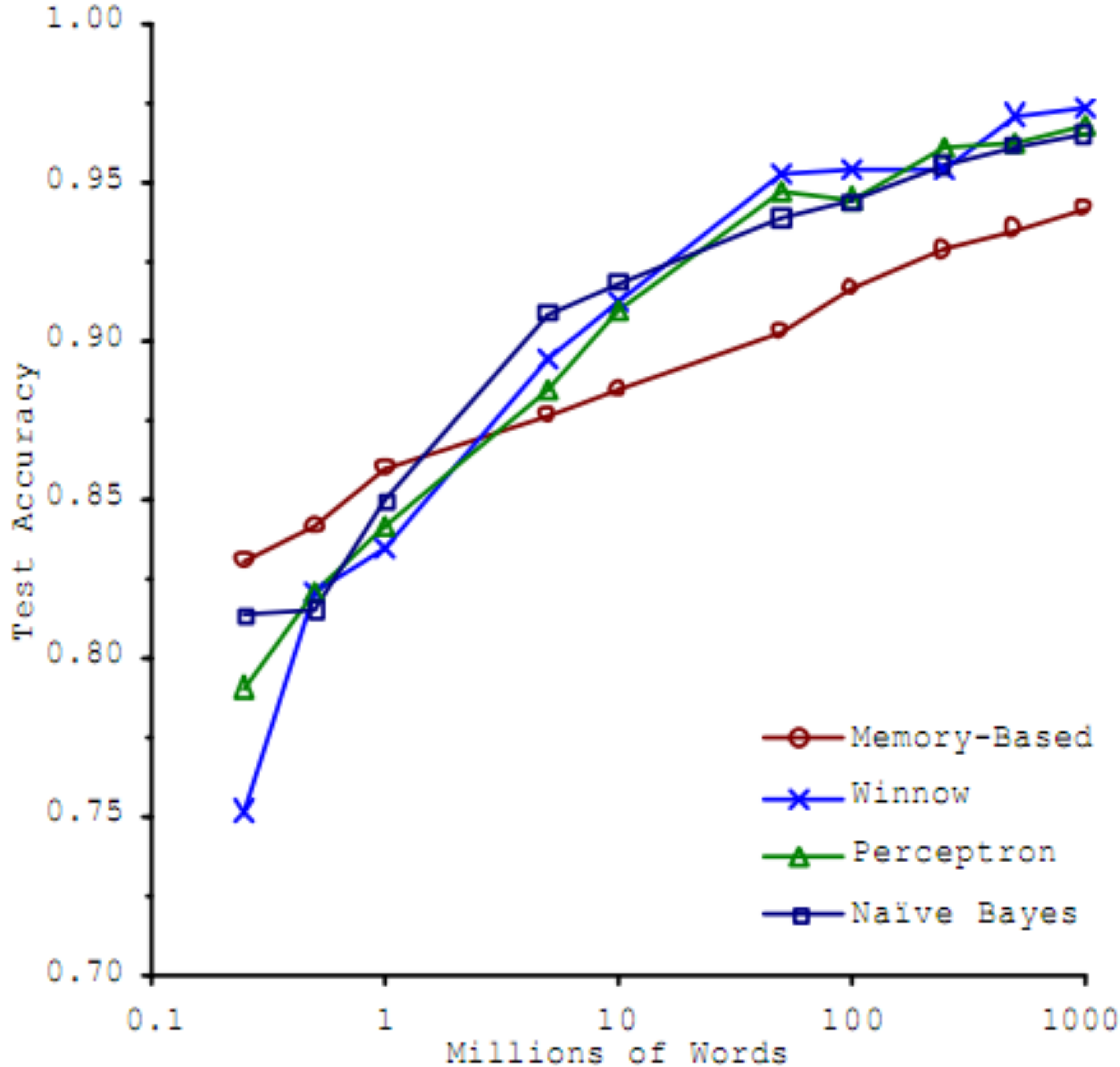
obs$_k$ = single object containing all items labeled as k

For each feature j

n_{kj} = # of occurrences of j in obs$_k$

$$p(j|k) = \frac{n_{kj}}{\sum_{j'} n_{kj'}}$$

Brill and Banko (2001)
With enough data, the classifier may not matter



Adapted from Jurafsky & Martin (draft)

Summary: Naïve Bayes is Not So Naïve, but not without issue

Pro

Very Fast, low storage requirements

Robust to Irrelevant Features

Very good in domains with many equally important features

Optimal if the independence assumptions hold

Dependable baseline for text classification (but often not the best)

Con

Model the posterior in one go? (e.g., use conditional maxent)

Are the features really uncorrelated?

Are plain counts always appropriate?

Are there “better” ways of handling missing/noisy data? (automated, more principled)

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Joint probability factorizes based on cliques in the graph

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Common name: **Markov Random Fields**

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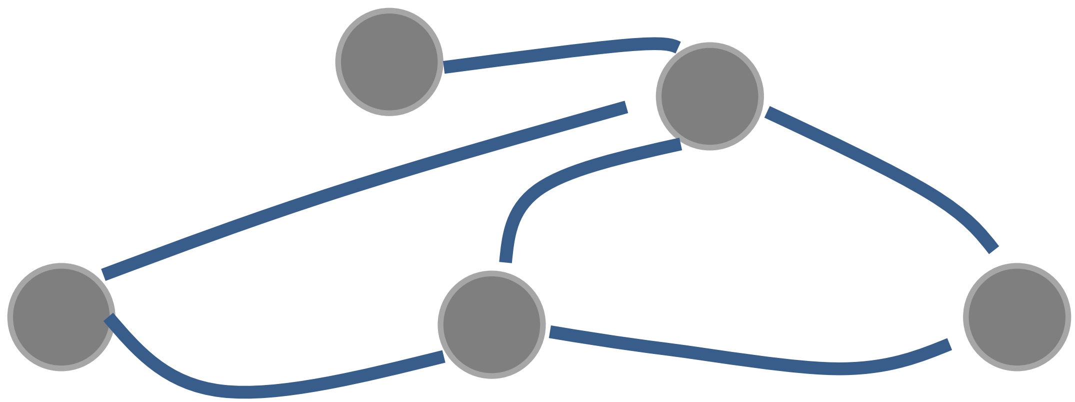
Joint probability factorizes based on cliques in the graph

Common name: **Markov Random Fields**

Undirected graphs can have an alternative formulation as **Factor Graphs**

Markov Random Fields: Undirected Graphs

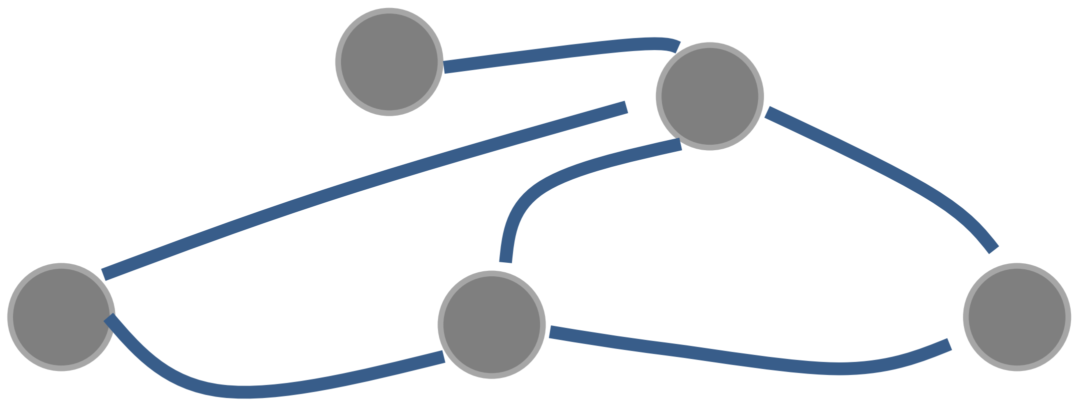


$$p(x_1, x_2, x_3, \ldots, x_N)$$

Markov Random Fields: Undirected Graphs

clique: subset of nodes, where nodes are pairwise connected

maximal clique: a clique that cannot add a node and remain a clique



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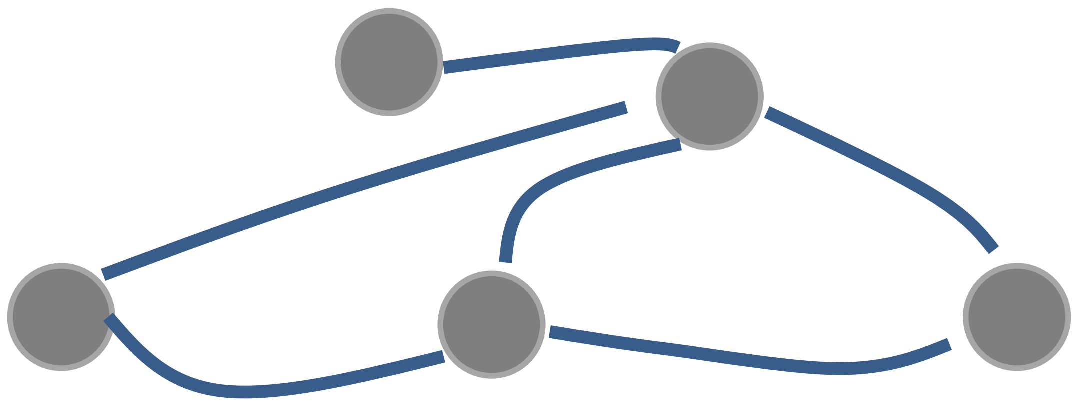

$$p(x_1, x_2, x_3, \ldots, x_N) = \frac{1}{Z} \prod_{C} \psi_C(x_c)$$

global normalization

maximal cliques

potential function (not necessarily a probability!)

variables part of the clique C

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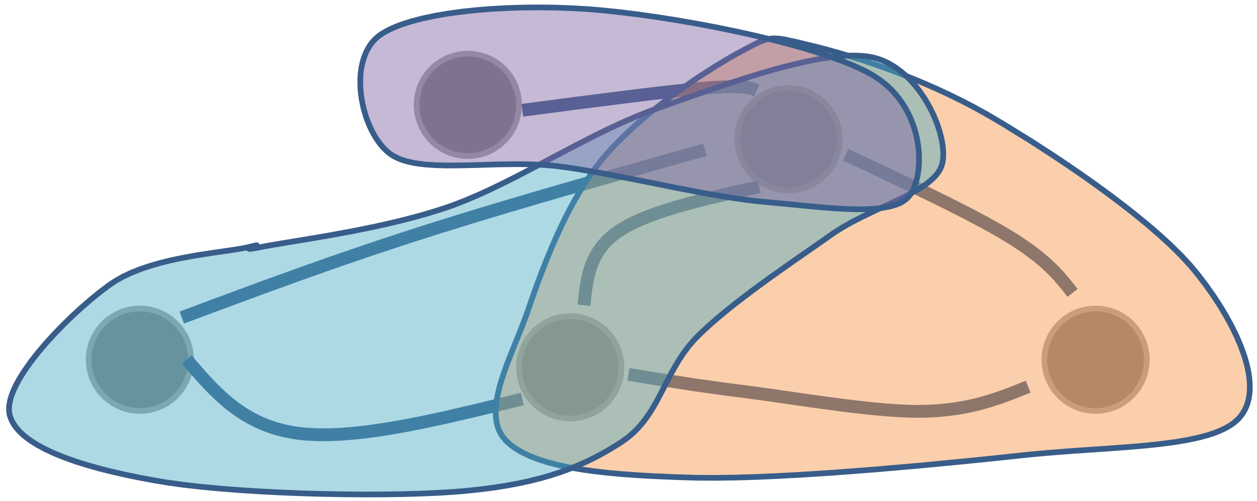

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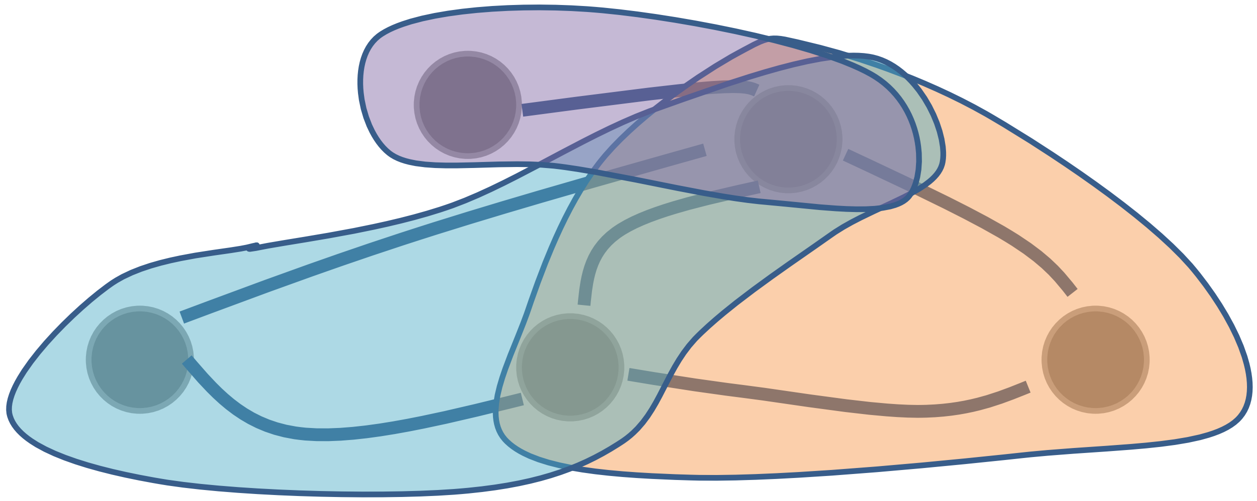

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Q: What restrictions should we place on the potentials ψ_C?

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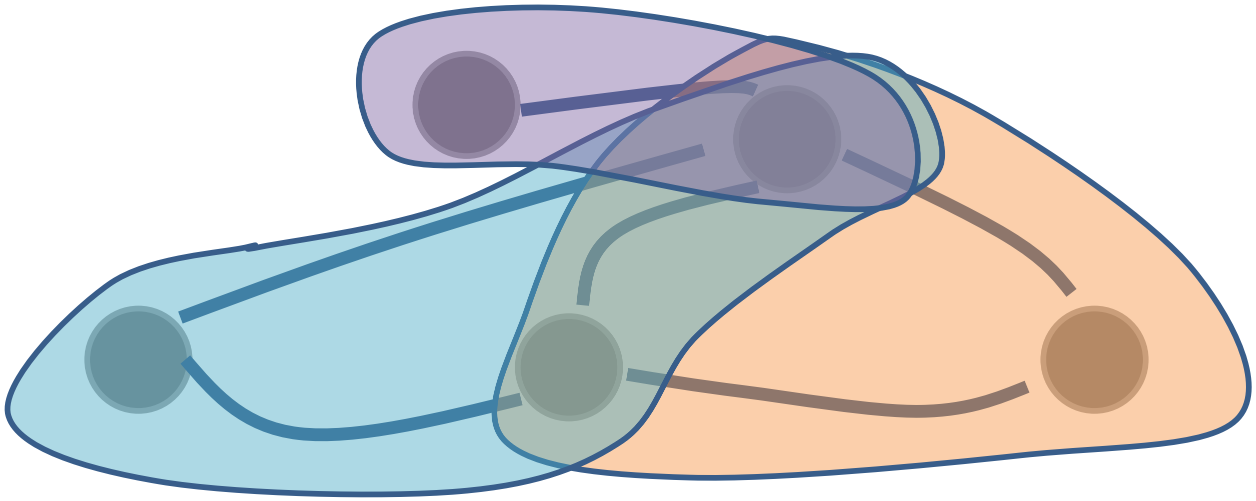

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Q: What restrictions should we place on the potentials ψ_C?

A: $\psi_C \geq 0$ (or $\psi_C > 0$)

Terminology: Potential Functions

$$p(x_1, x_2, x_3, \ldots, x_N) = \frac{1}{Z}\prod_C \psi_C(x_c)$$

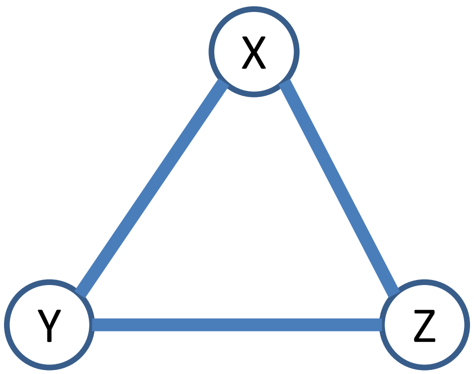
energy function (for clique C)

(get the total energy of a configuration by summing the individual energy functions)

$$\psi_C(x_c) = \exp -E(x_C)$$

Boltzmann distribution

Ambiguity in Undirected Model Notation



$$p(x, y, z) \propto \psi(x, y, z)$$

- - - - - - - - - - - - - - - -

$$p(x, y, z) \propto \psi_{1(x,y)}\psi_{2(y,z)}\psi_{3(x,z)}$$

Outline

Directed Graphical Models

- Naïve Bayes

Undirected Graphical Models

- Factor Graphs
- Ising Model

Message Passing: Graphical Model Inference

MRFs as Factor Graphs

Undirected graphs: G=(V,E) that represents $p(X_1, \ldots, X_N)$

Factor graph of *p*: Bipartite graph of evidence nodes X, factor nodes F, and edges T

Evidence nodes X are the random variables

Factor nodes F take values associated with the *potential functions*

Edges show what variables are used in which factors

MRFs as Factor Graphs

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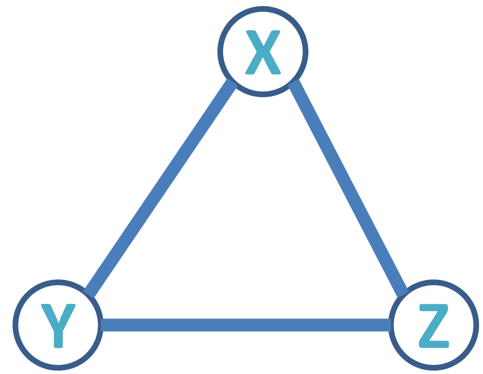
G=(V,E) that represents
$p(X_1, \ldots, X_N)$



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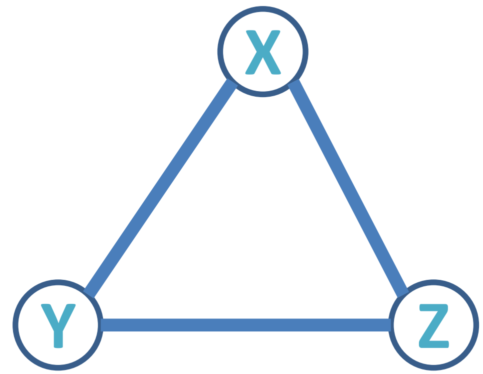

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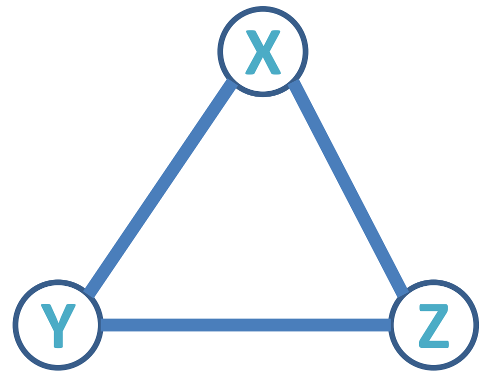

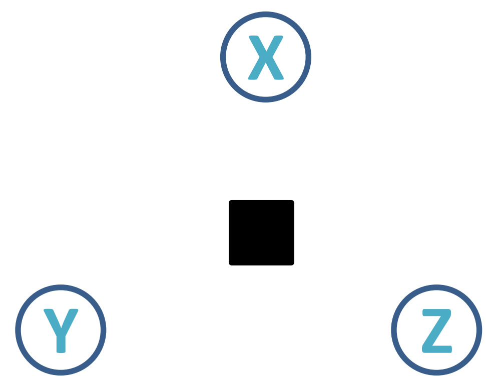

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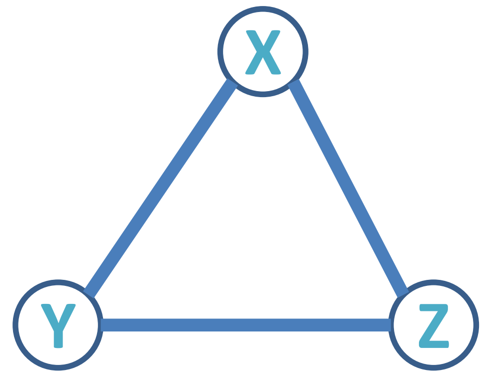
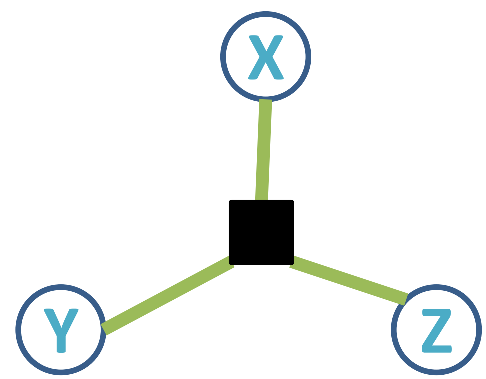
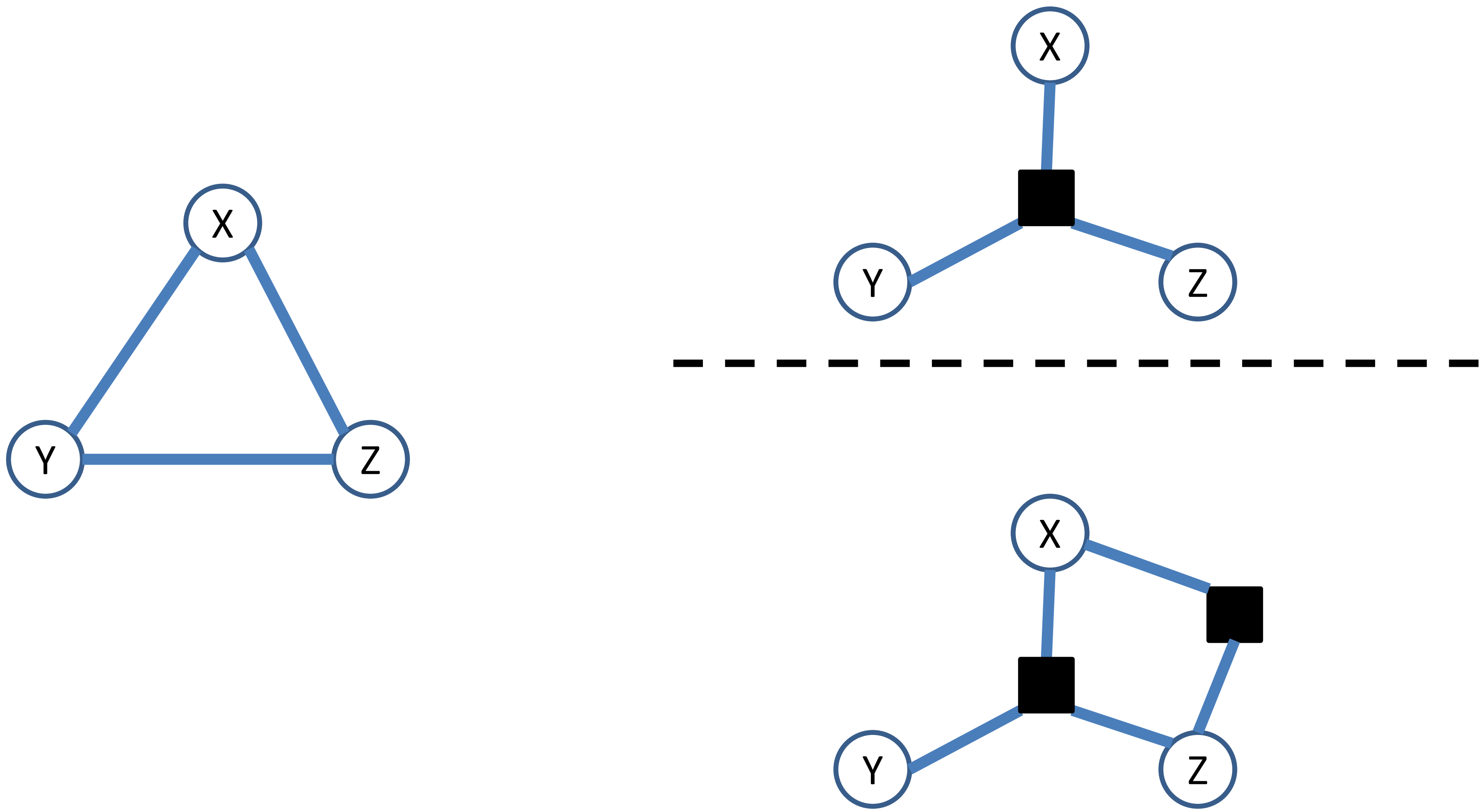


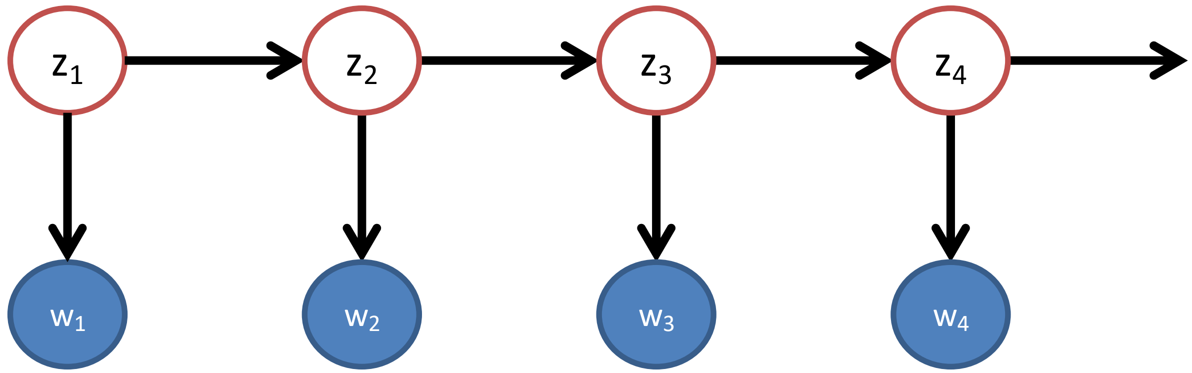


Different Factor Graph Notation for the Same Graph



Example: Linear Chain

Directed **(e.g., hidden Markov model [HMM]; generative)**



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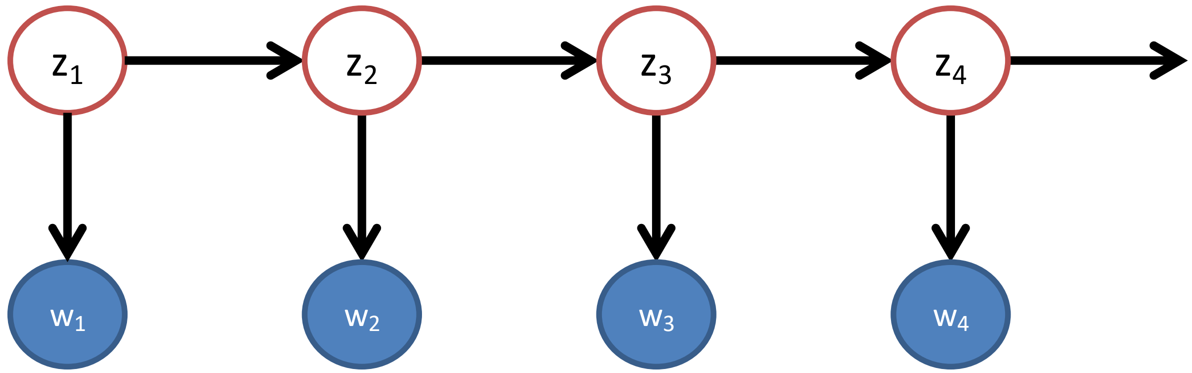

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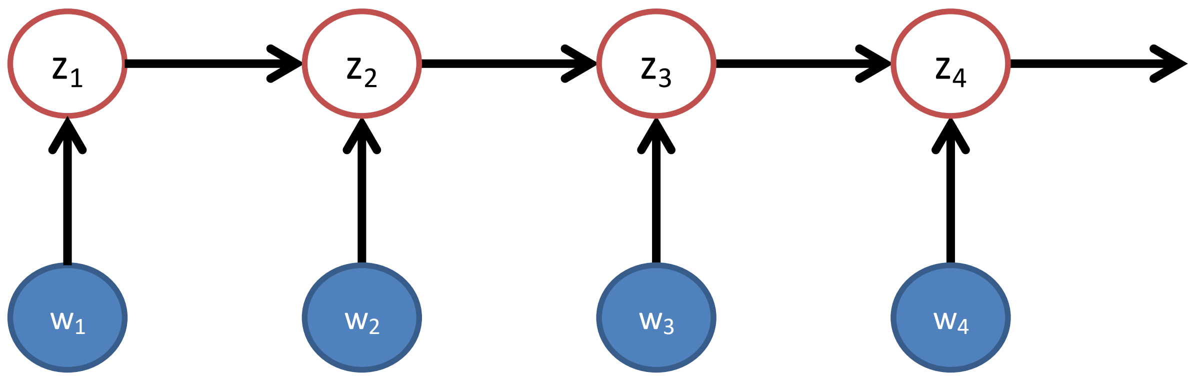

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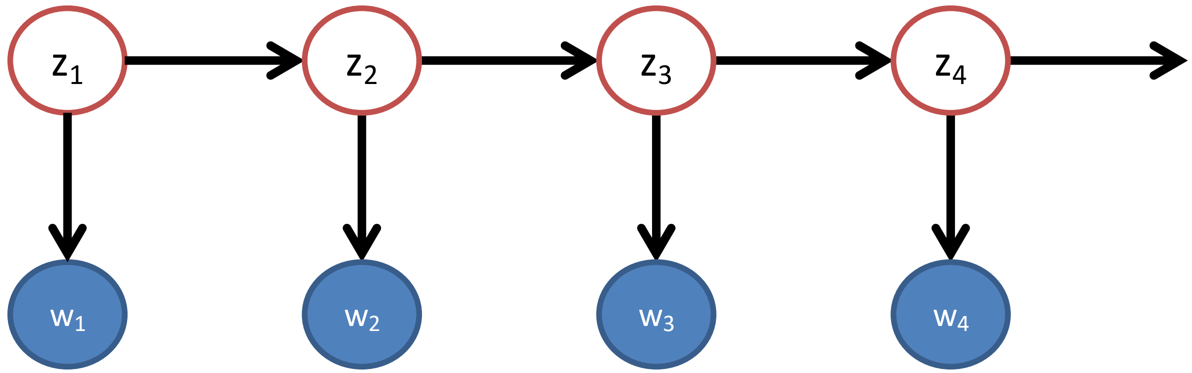

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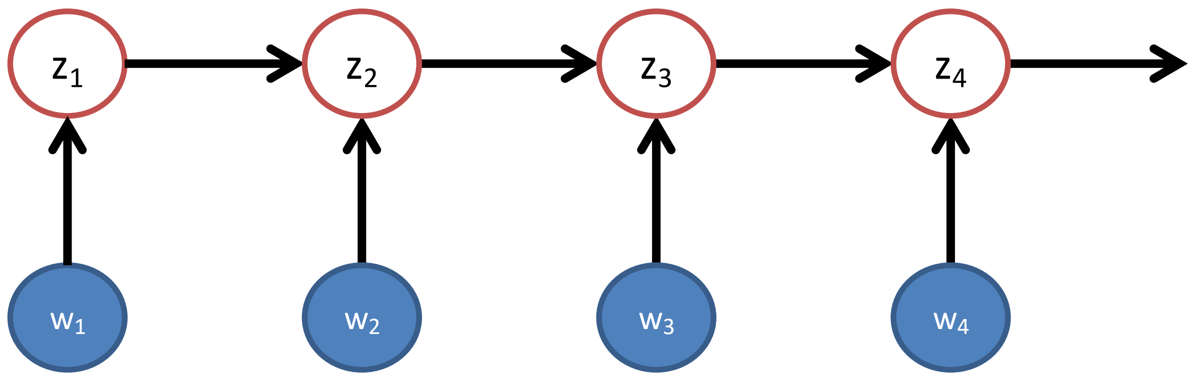

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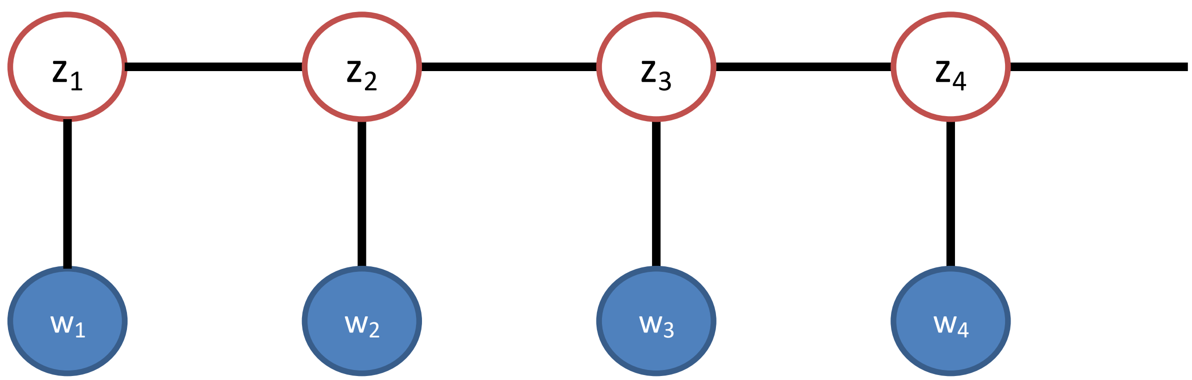

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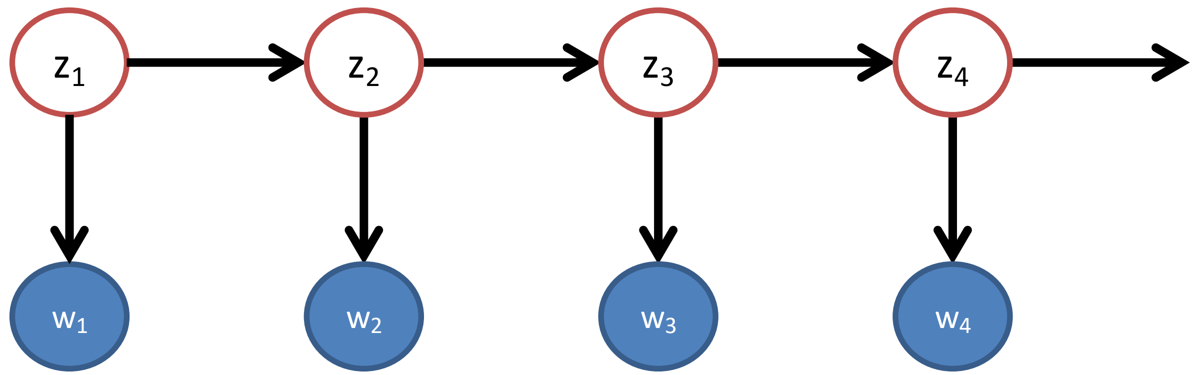

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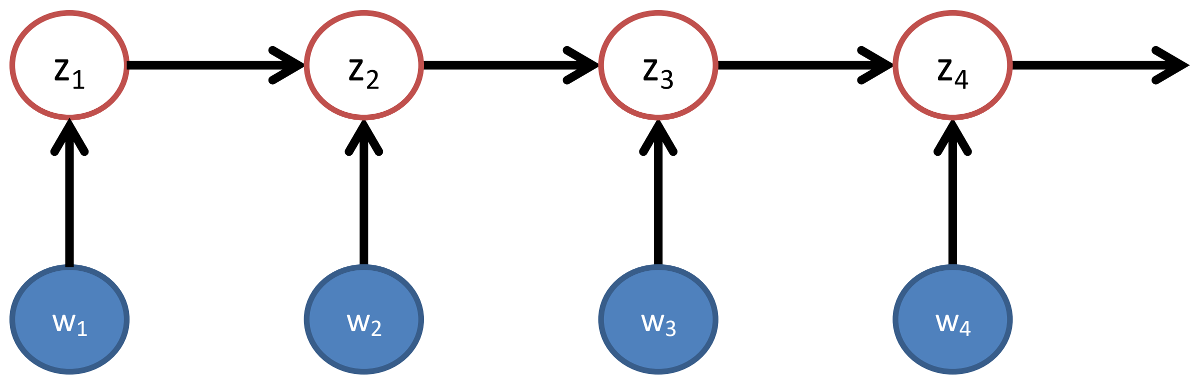
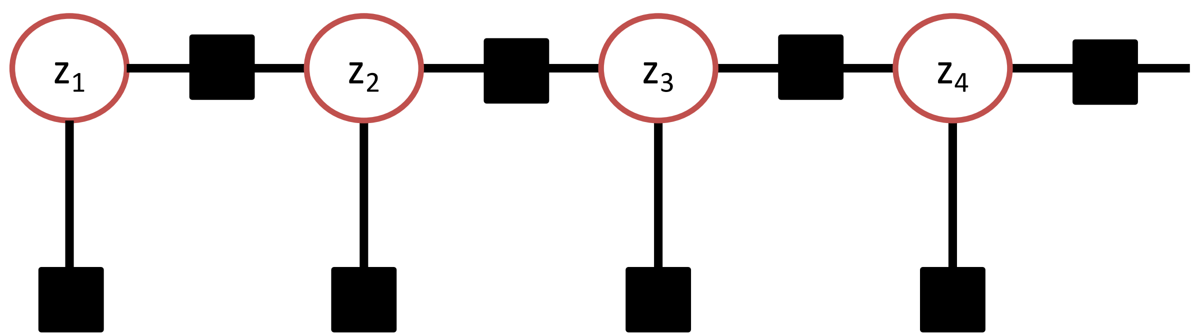
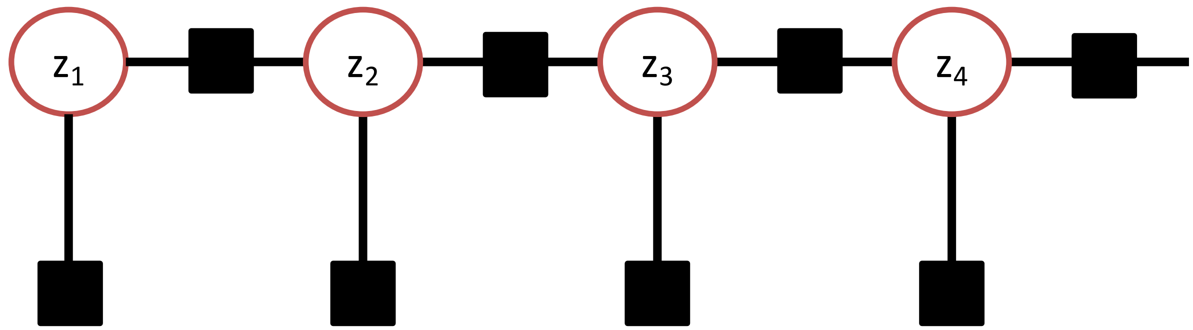


Undirected as factor graph (e.g., conditional random field [CRF])



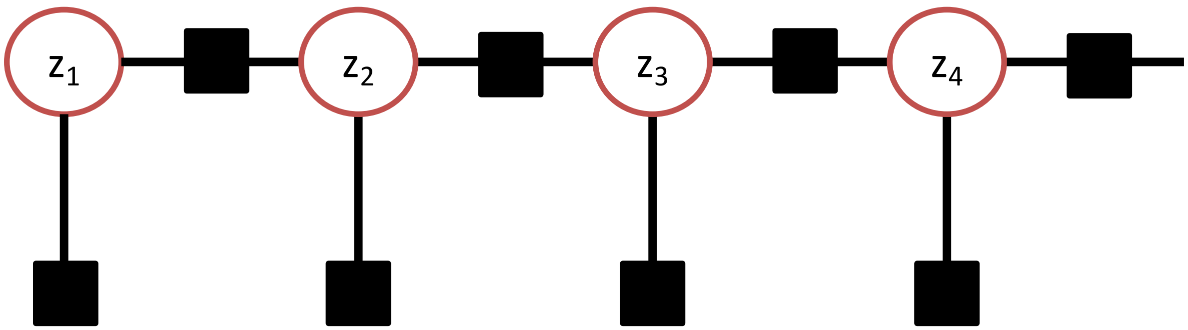
Example: Linear Chain Conditional Random Field



Widely used in applications like
part-of-speech tagging

Noun-Mod *Noun* *Verb* *Noun*
President Obama told Congress …

Example: Linear Chain Conditional Random Field



Widely used in applications like
part-of-speech tagging

Noun-Mod *Noun* *Verb* *Noun*
President Obama told Congress …

and named entity recognition

Person *Person* *Other* *Org.*
President Obama told Congress …

Linear Chain CRFs for Part of Speech Tagging

A linear chain CRF is a conditional probabilistic model of the sequence of tags $z_1, z_2, \dots, z_N$ conditioned on the *entire* input sequence $x_{1:N}$

Linear Chain CRFs for Part of Speech Tagging

$$p(\clubsuit | \Diamond)$$

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Linear Chain CRFs for Part of Speech Tagging

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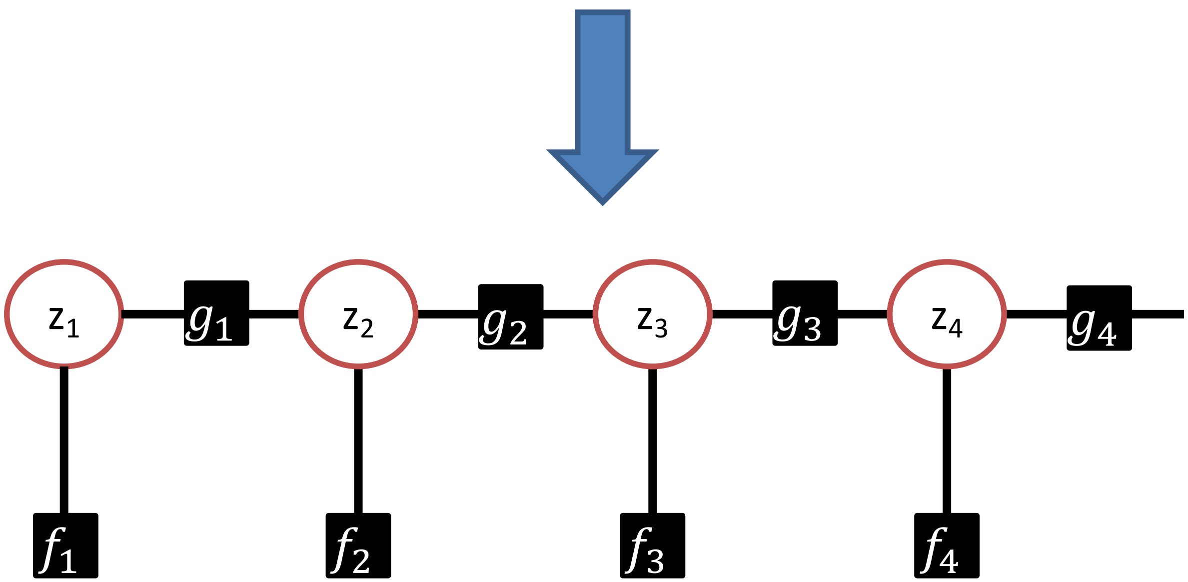
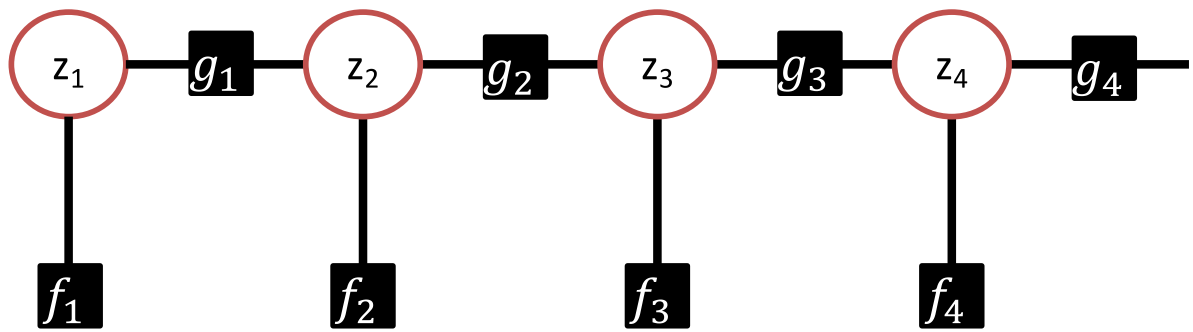


Linear Chain CRFs for Part of Speech Tagging

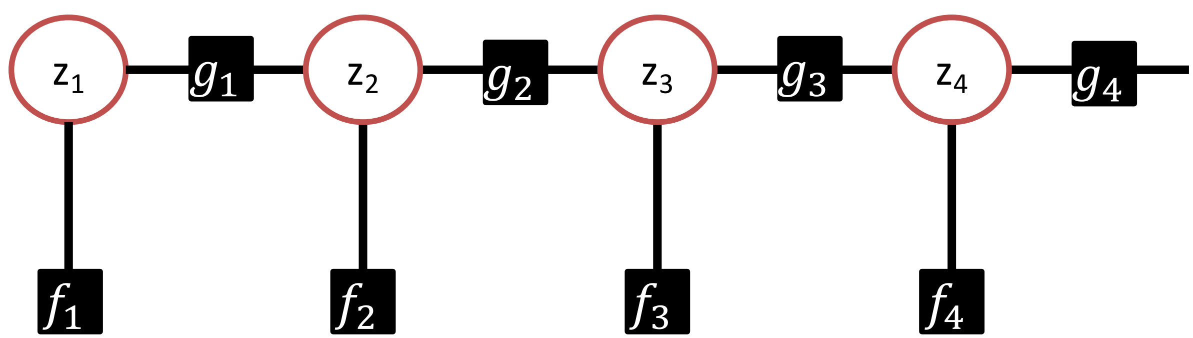


$$p(z_1, z_2, \ldots, z_N | x_{1:N}) \propto$$

$$\prod_{i=1}^{N} \exp(\langle \theta^{(f)}, f_i(z_i) \rangle + \langle \theta^{(g)}, g_i(z_i, z_{i+1}) \rangle)$$

Linear Chain CRFs for Part of Speech Tagging

g_j: inter-tag features (can depend on any/all input words $x_{1:N}$)



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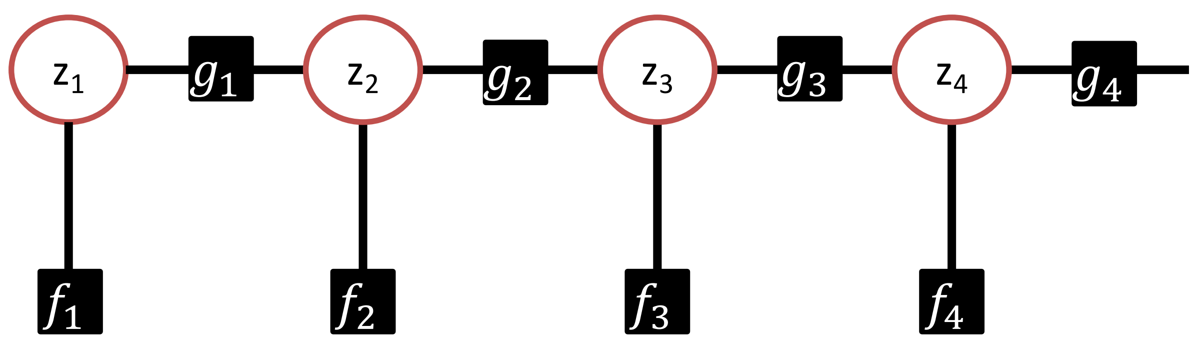

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g_j: inter-tag features (can depend on any/all input words $x_{1:N}$)

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Feature design, just like in maxent models!

Linear Chain CRFs for Part of Speech Tagging

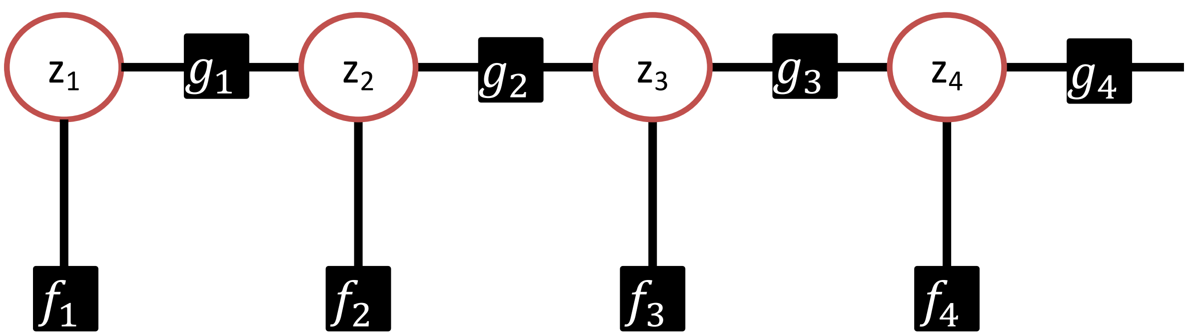
g_j: inter-tag features (can depend on any/all input words $x_{1:N}$)

f_i: solo tag features (can depend on any/all input words $x_{1:N}$)

Example:

$$g_{j,N\to V}(\mathrm{z_j}, \mathrm{z_{j+1}}) = 1 \text{ (if } \mathrm{z_j} == \mathrm{N} \;\&\; \mathrm{z_{j+1}} == \mathrm{V}\text{) else } 0$$

$$g_{j,\mathrm{told},N\to V}(\mathrm{z_j}, \mathrm{z_{j+1}}) = 1 \text{ (if } \mathrm{z_j} == \mathrm{N} \;\&\; \mathrm{z_{j+1}} == \mathrm{V} \;\&\; \mathrm{x_j} == \mathrm{told}\text{) else } 0$$



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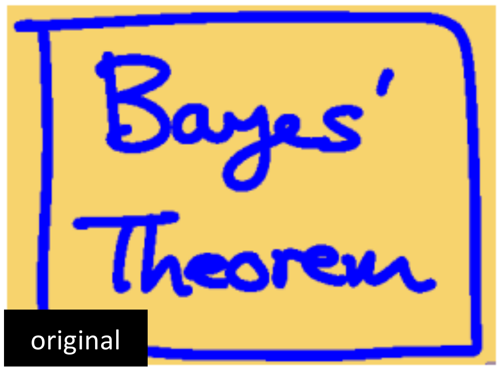
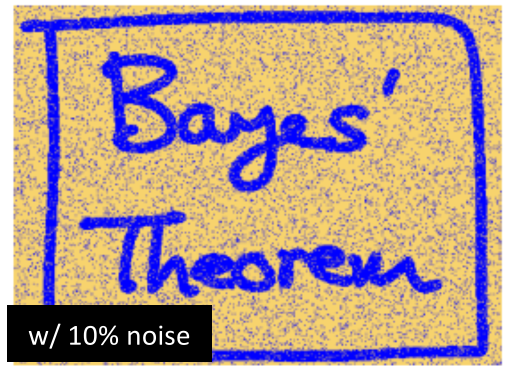
Example: Ising Model

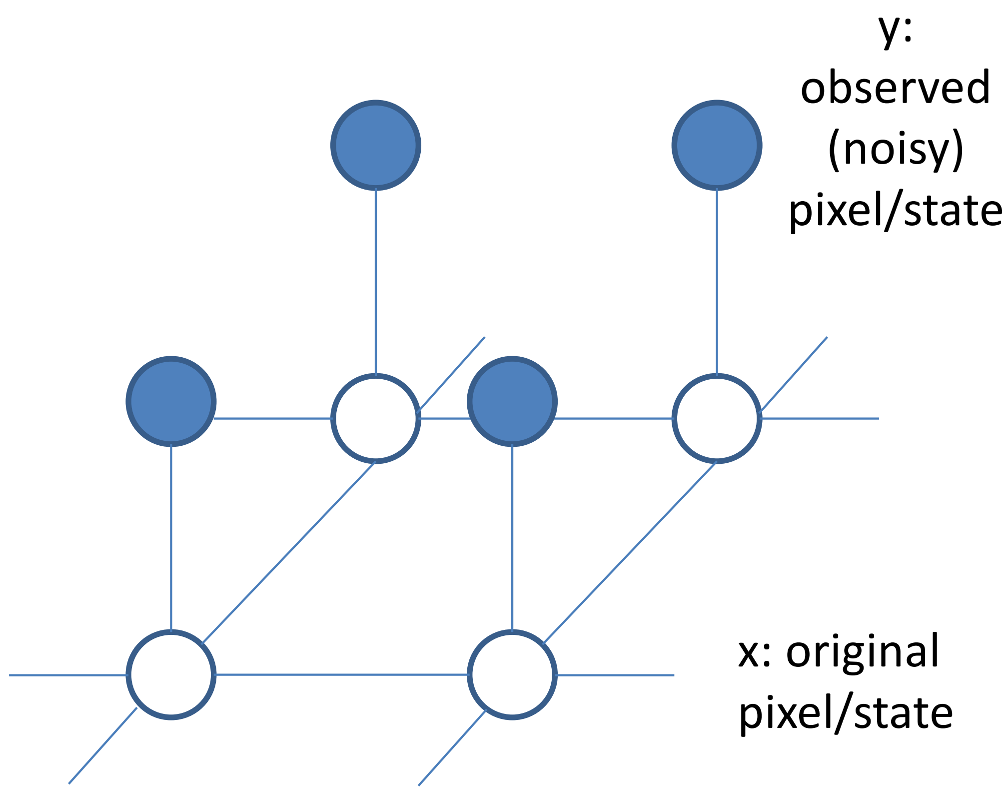
Image denoising (Bishop, 2006; Fig 8.30)



X



Y



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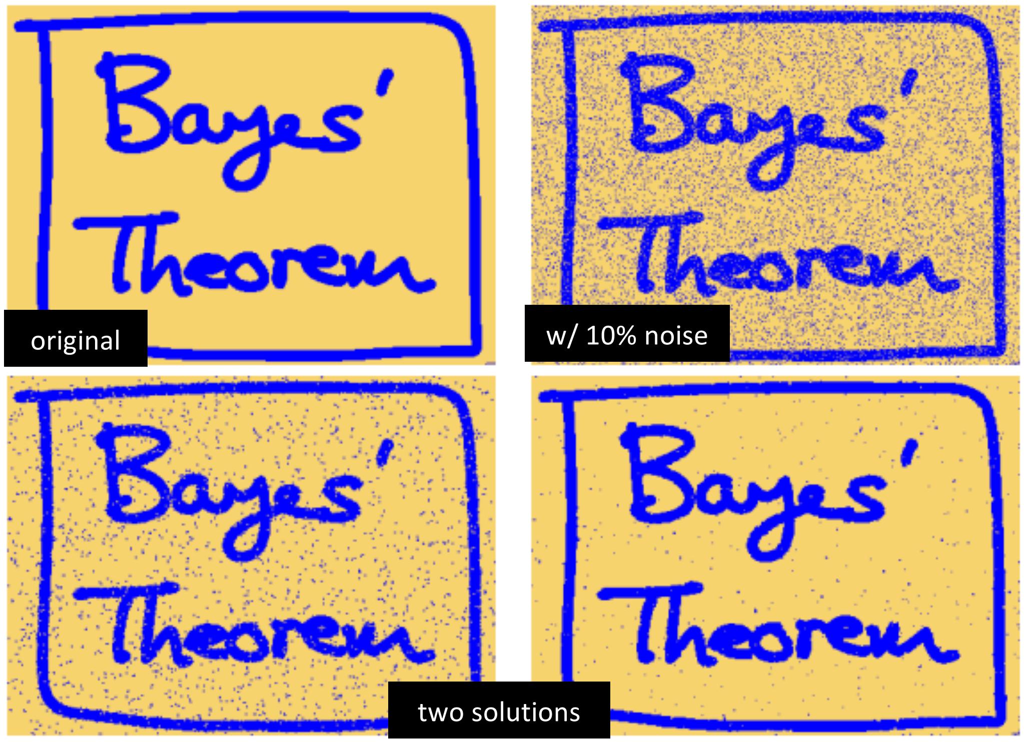

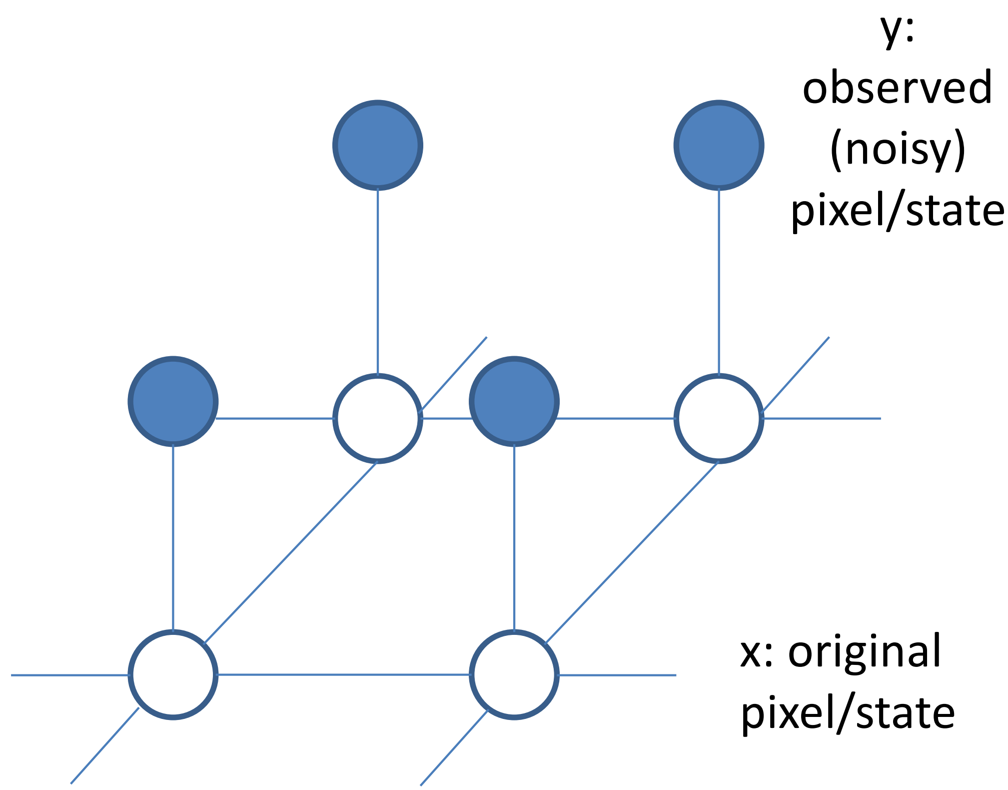

Q: What are the cliques?

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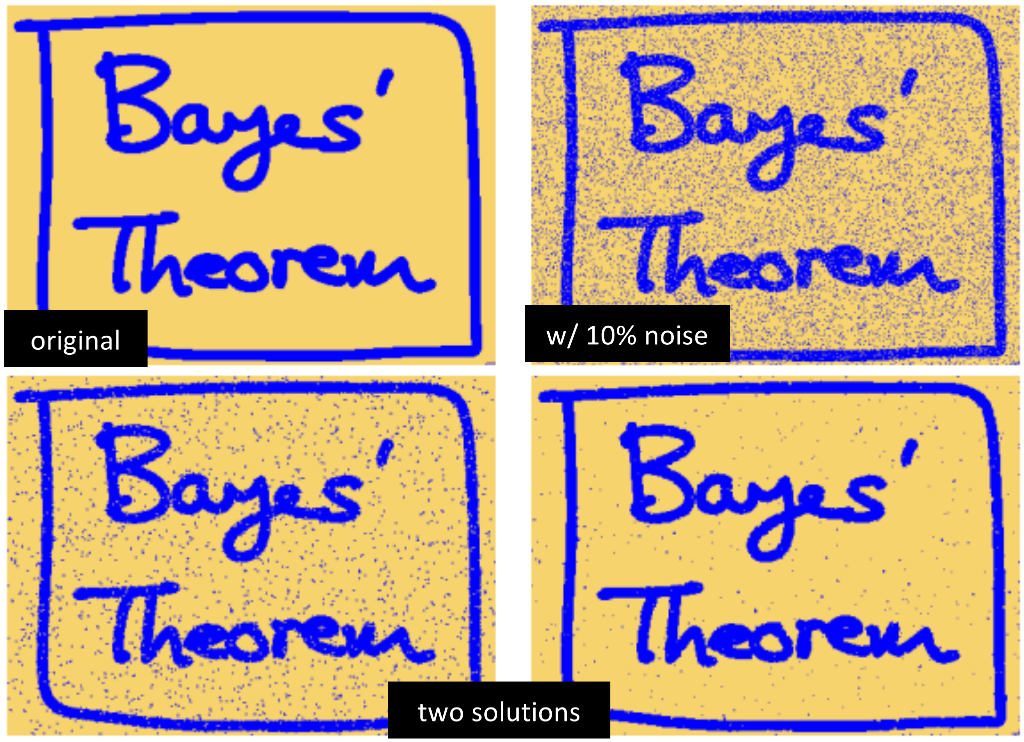

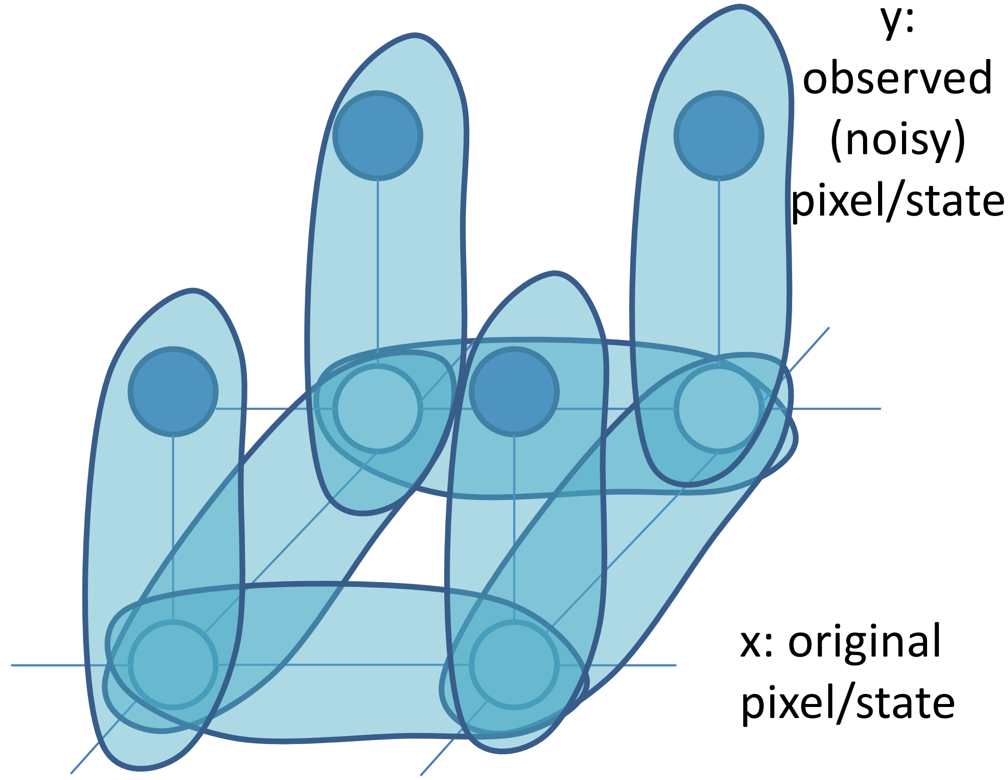

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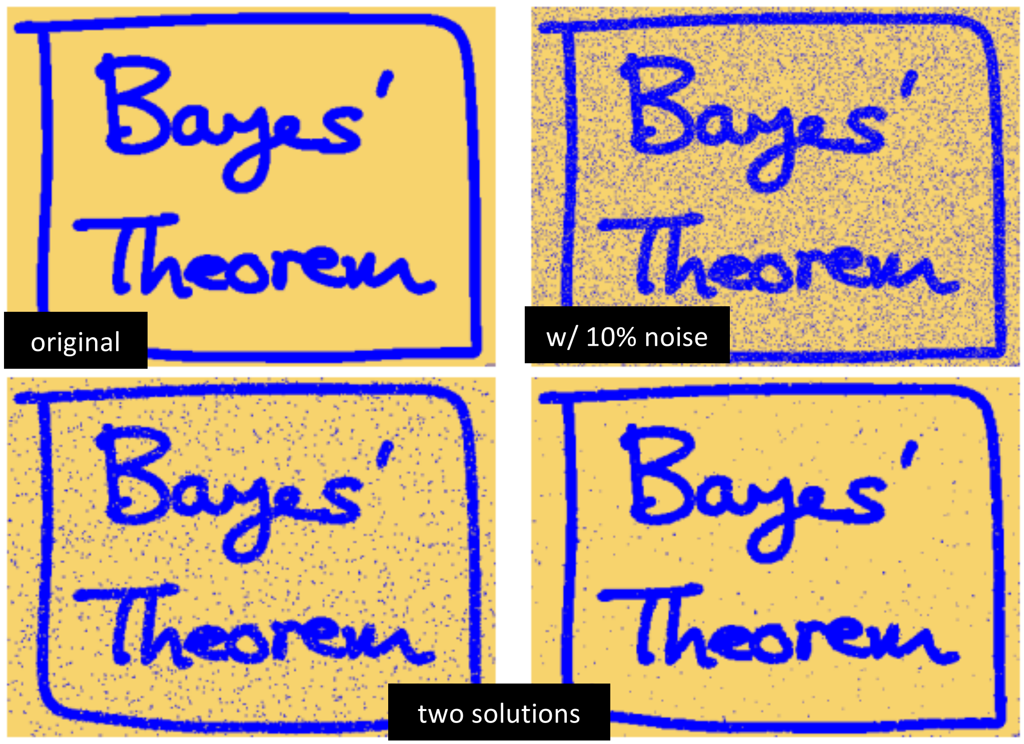

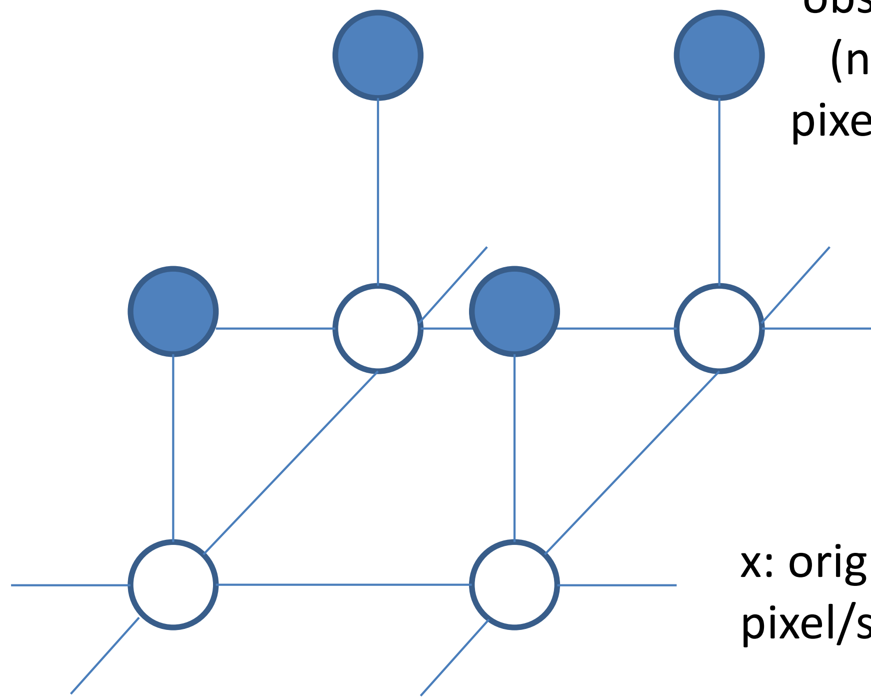

y: observed (noisy) pixel/state

x: original pixel/state

neighboring pixels should be similar

$$E(x,y) = h\sum_i x_i - \beta \sum_{ij} x_i x_j - \eta \sum_i x_i y_i$$

allow for a bias

x_i and y_i should be correlated

Example: Ising Model

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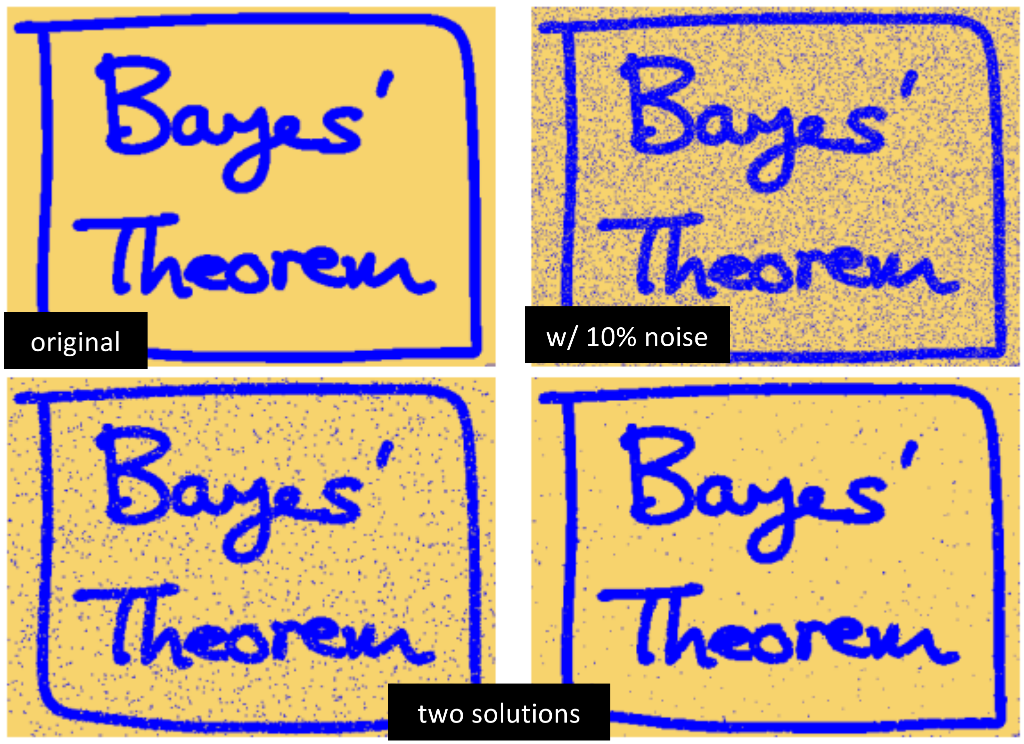

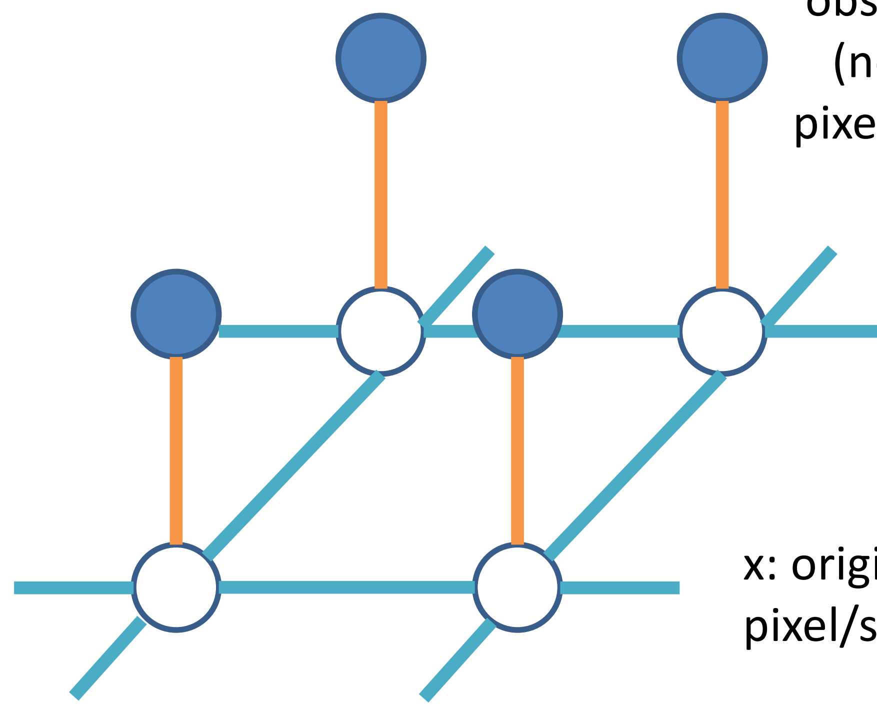

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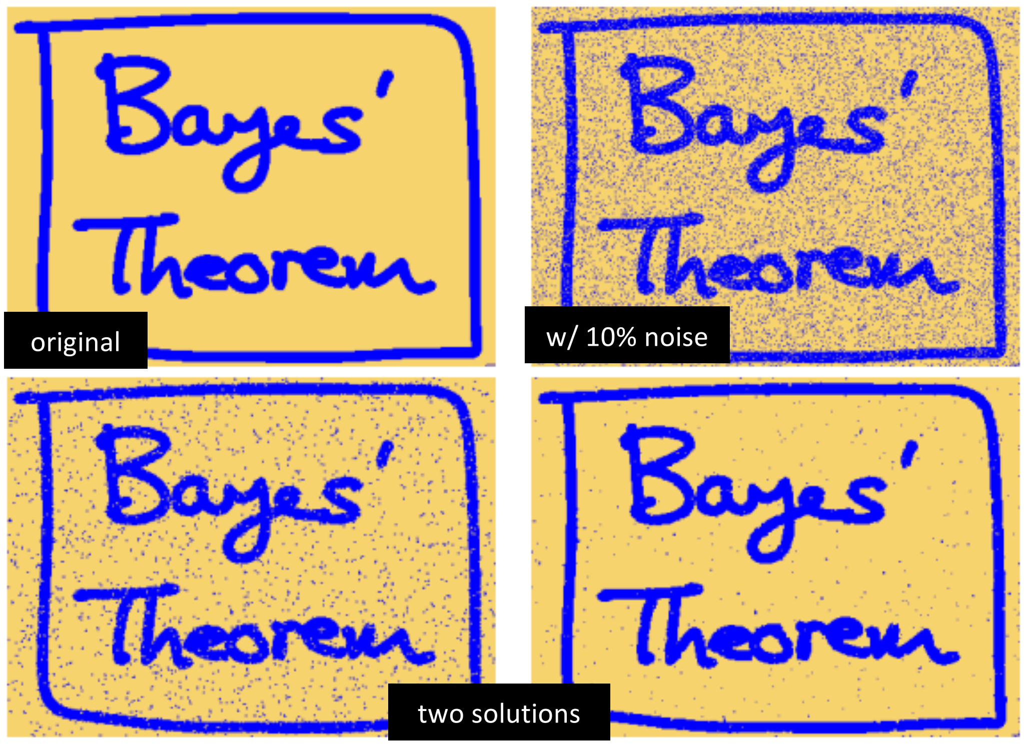

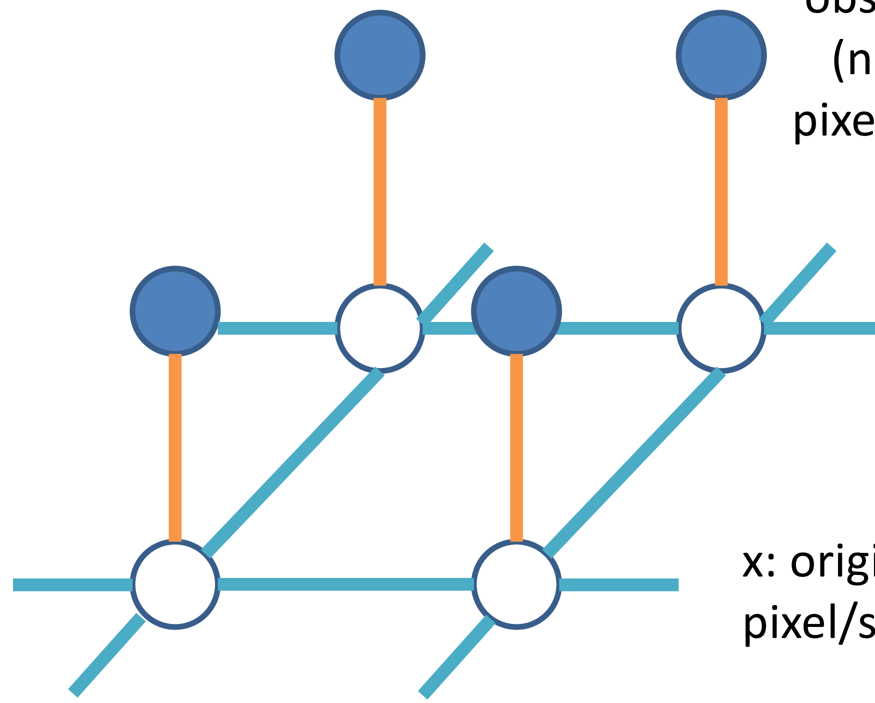

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Q: Why subtract β and η?

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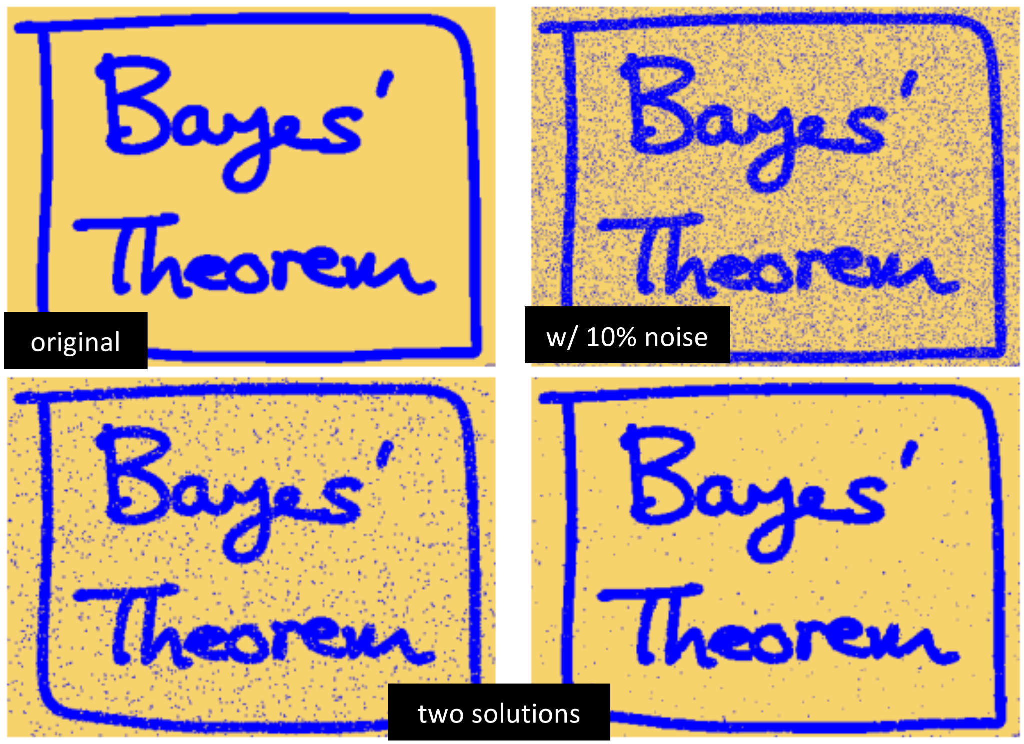

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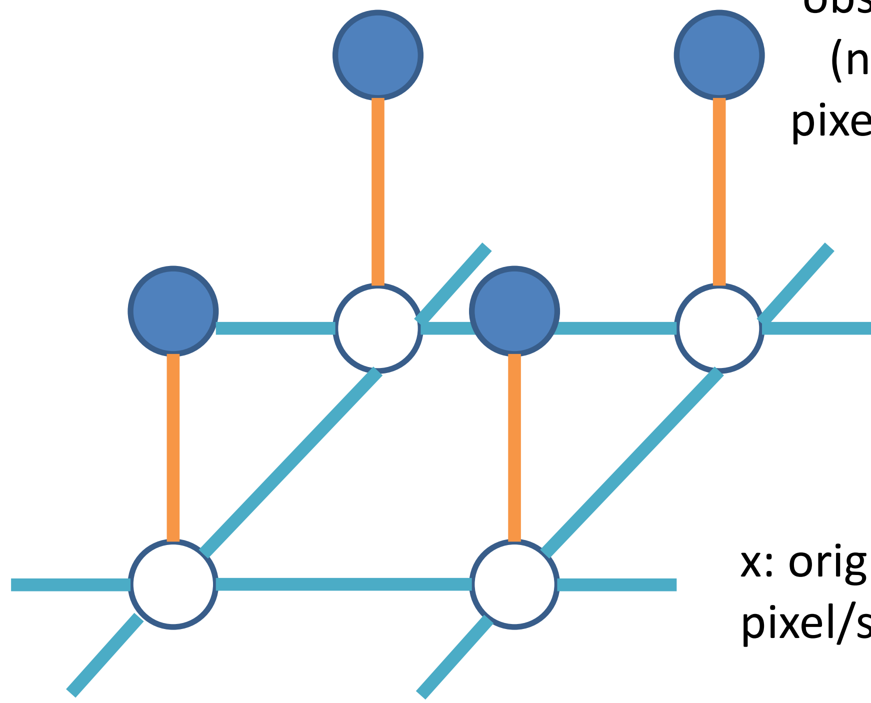

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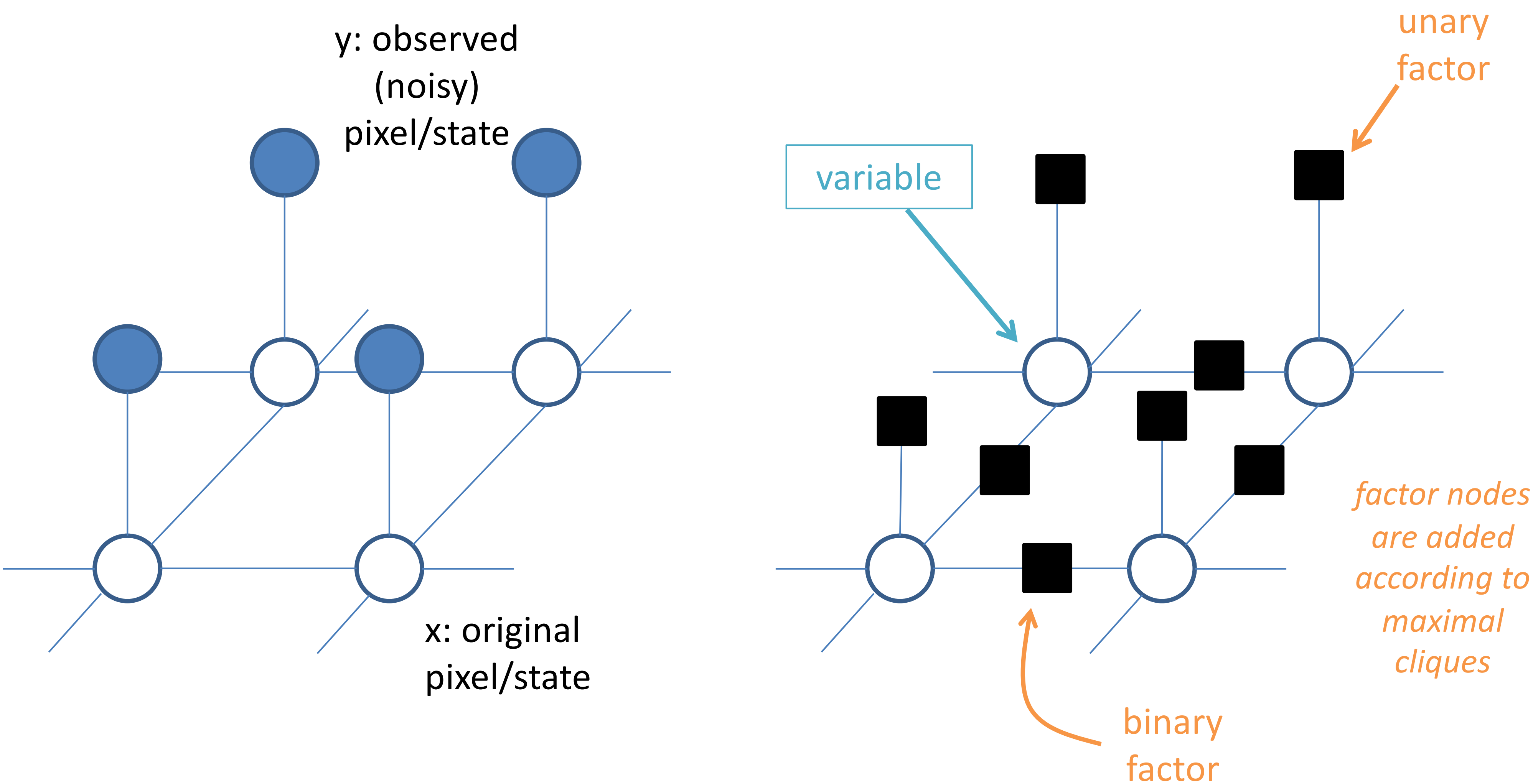
allow for a bias

x_i and y_i should be correlated

A: Better states → lower energy (higher potential)
$\psi_C(x_c) = \exp -E(x_C)$

Markov Random Fields with **Factor Graph** Notation



factor graphs are **bipartite**

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Two Problems for Undirected Models

$$p(x_1, x_2, x_3, \dots, x_N) = \frac{1}{Z}\prod_{C} \psi_C(x_c)$$

Finding the normalizer

Computing the marginals

Two Problems for Undirected Models

$$p(x_1, x_2, x_3, \dots, x_N) = \frac{1}{Z} \prod_C \psi_C(x_c)$$

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$$Z = \sum_x \prod_c \psi_c(x_c)$$

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Computing the marginals

Sum over all variable combinations, with the x_n coordinate fixed

$$Z_n(v) = \sum_{x: x_n = v} \prod_c \psi_c(x_c)$$

Example: 3 variables, fix the 2nd dimension

$$Z_2(v) = \sum_{x_1} \sum_{x_3} \prod_c \psi_c(x = (x_1, v, x_3))$$

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A: Many different combinations

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Message Passing: Count the Soldiers

If you are the front soldier in the line, say the number 'one' to the soldier behind you.

If you are the rearmost soldier in the line, say the number 'one' to the soldier in front of you.

If a soldier ahead of or behind you says a number to you, add one to it, and say the new number to the soldier on the other side

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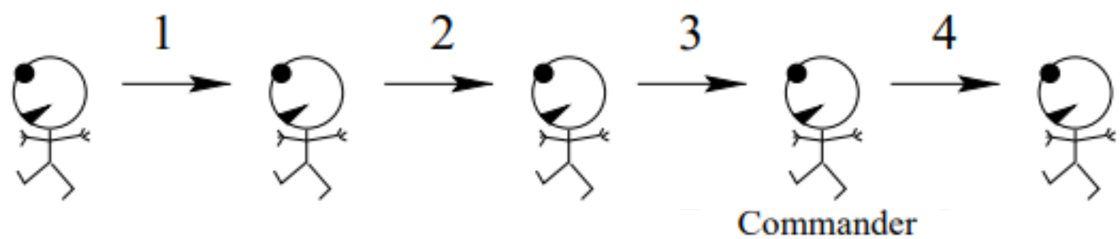

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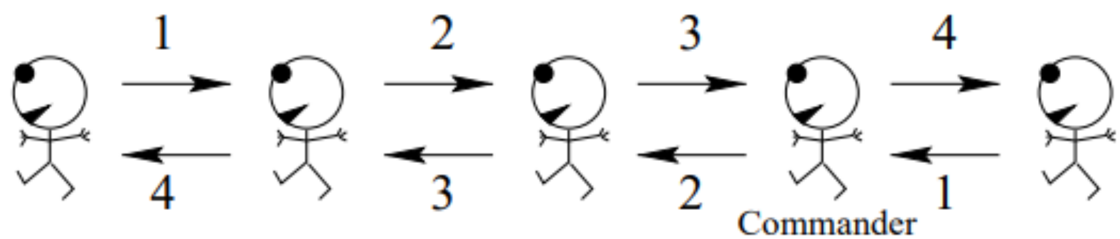


Sum-Product Algorithm

Main idea: message passing

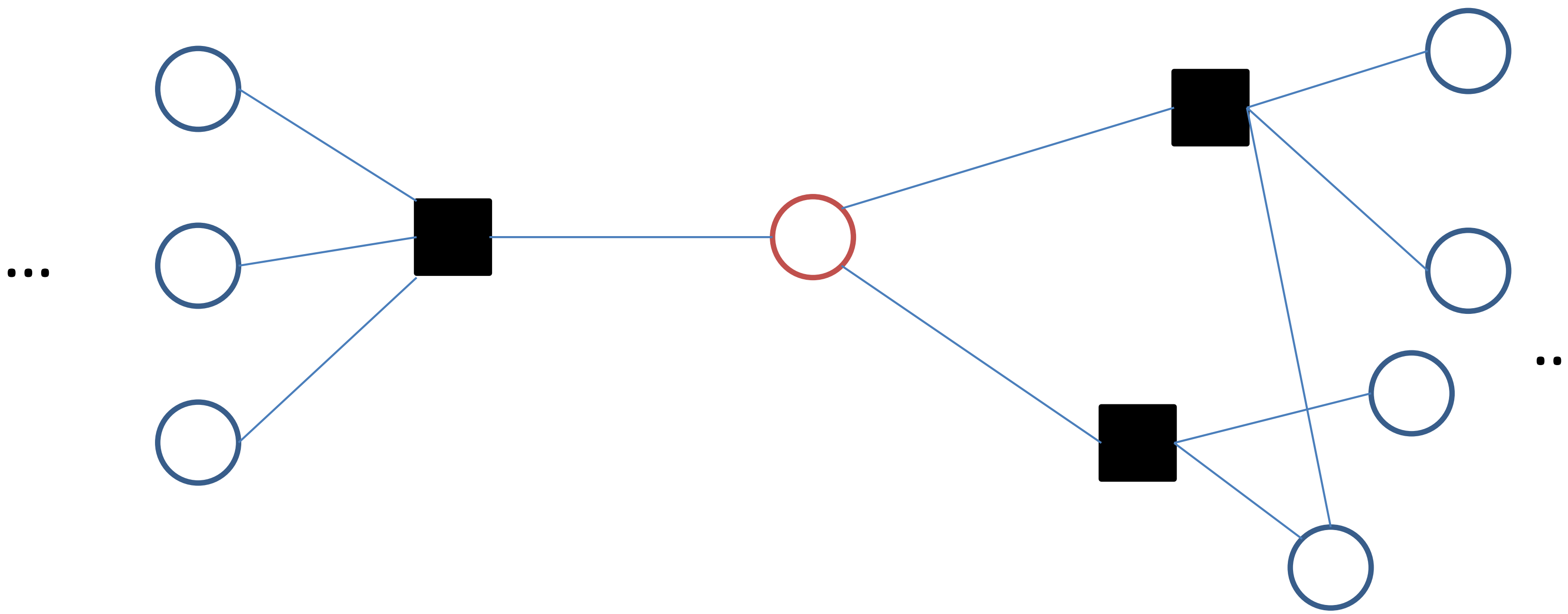
An exact inference algorithm for tree-like graphs

Belief propagation (forward-backward for HMMs) is a special case

Sum-Product

definition of marginal

$$p(x_i = v) = \prod_{x:x_i=v} p(x_1, x_2, \ldots, x_i, \ldots, x_N)$$



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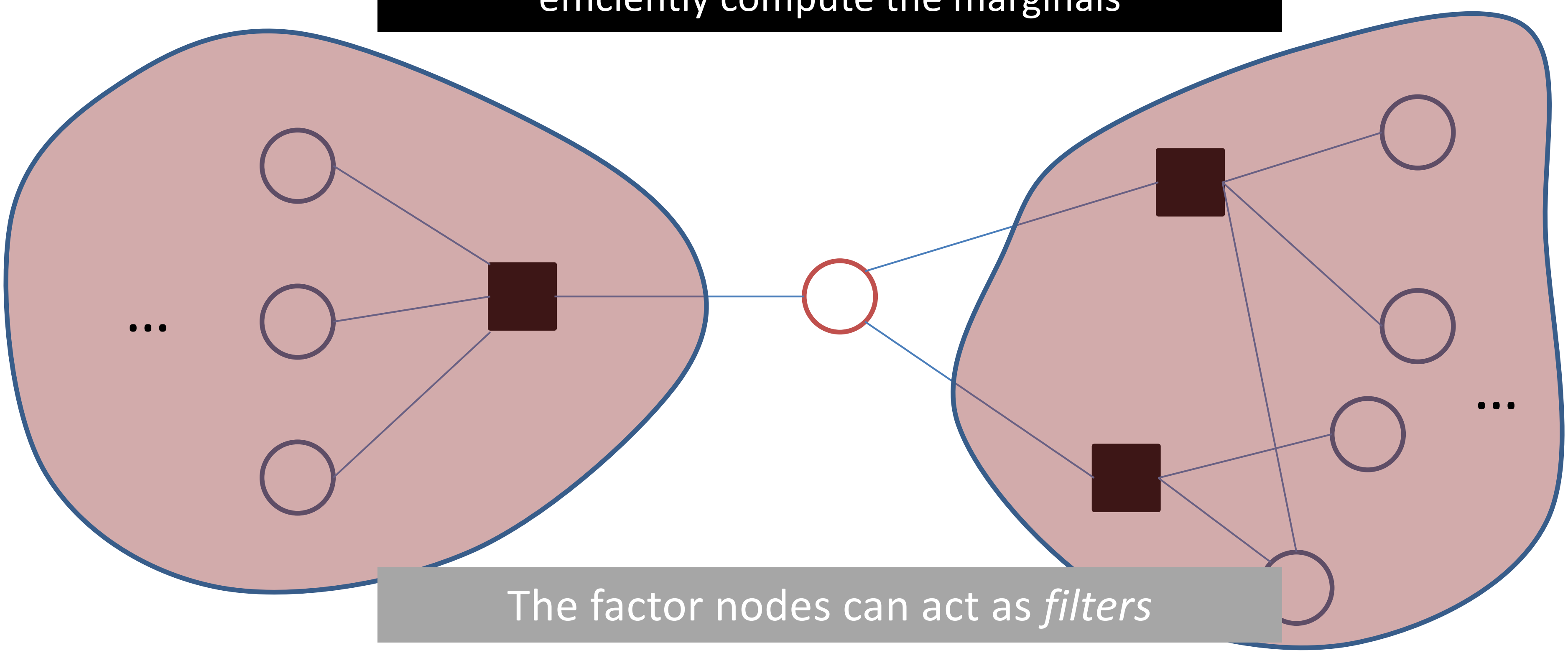
definition of marginal

$$p(x_i = v) = \prod_{x:x_i=v} p(x_1, x_2, \dots, x_i, \dots, x_N)$$

main idea: use bipartite nature of graph to efficiently compute the marginals



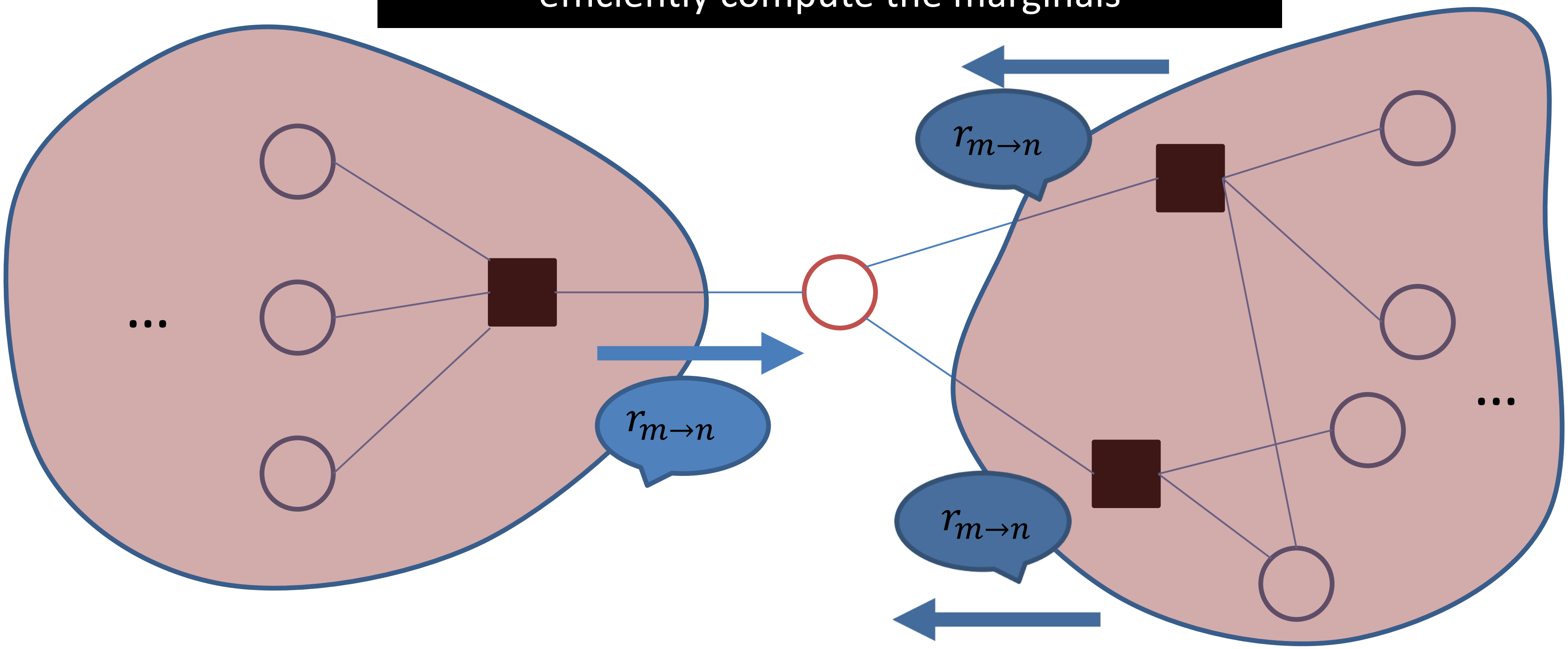
The factor nodes can act as *filters*

Sum-Product

definition of marginal

$$p(x_i = v) = \prod_{x:x_i=v} p(x_1, x_2, \dots, x_i, \dots, x_N)$$

main idea: use bipartite nature of graph to efficiently compute the marginals

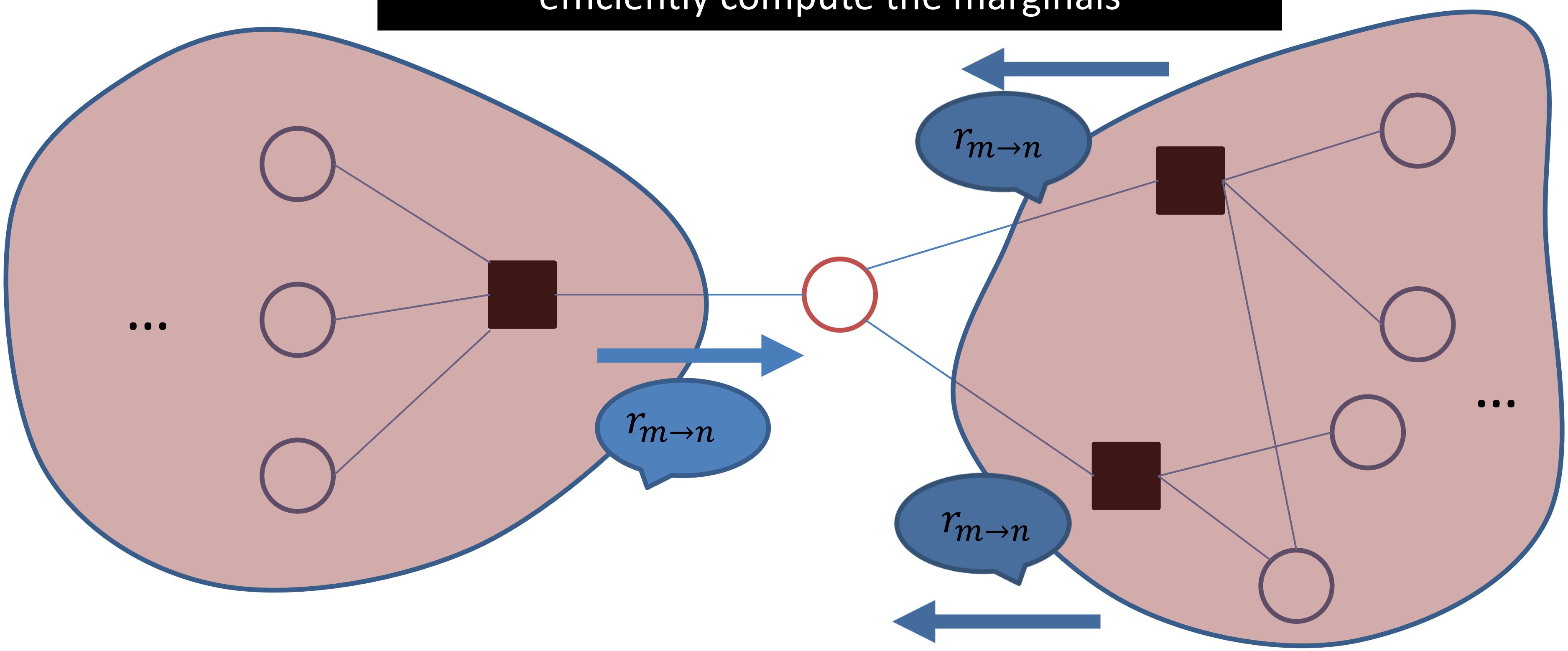


Sum-Product

alternative marginal computation

$$p(x_i = v) = \prod_f r_{f \to x_i}(x_i)$$

main idea: use bipartite nature of graph to efficiently compute the marginals



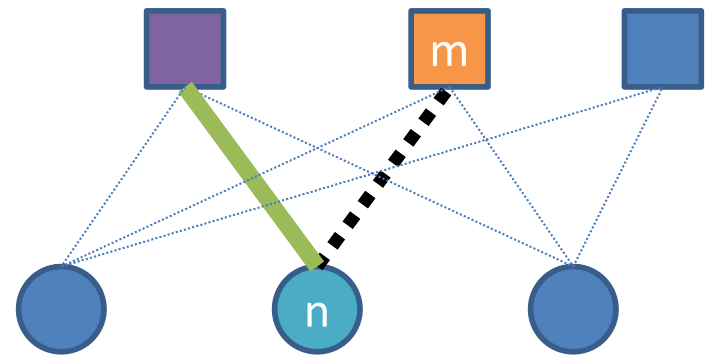
Sum-Product

From variables to factors

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \backslash m} r_{m' \to n}(x_n)$$

set of factors in which variable n participates

default value of 1 if empty product

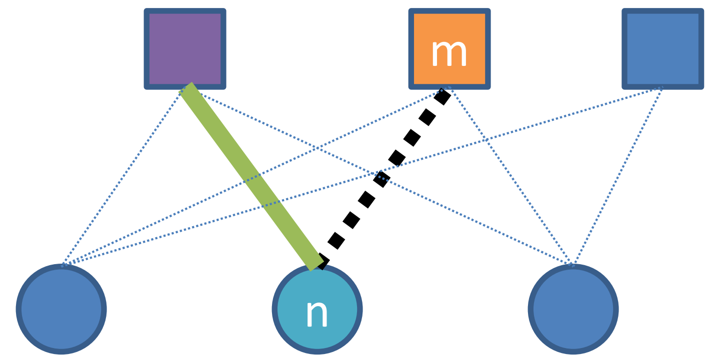


Sum-Product

From variables to factors

$$\underline{q_{n\to m}}(x_n) = \prod_{m' \in M(n)\setminus m} r_{m'\to n}(x_n)$$

set of factors in which variable n participates



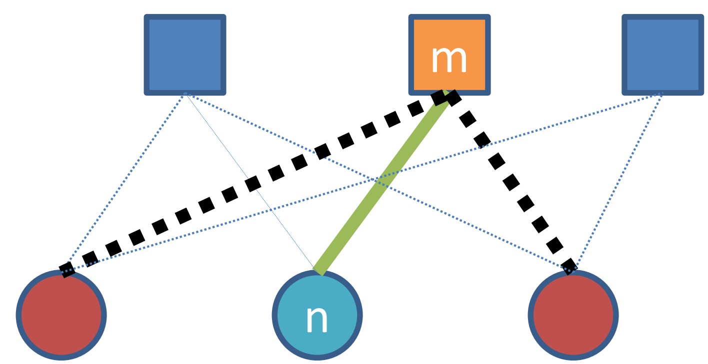
default value of 1 if empty product

From factors to variables

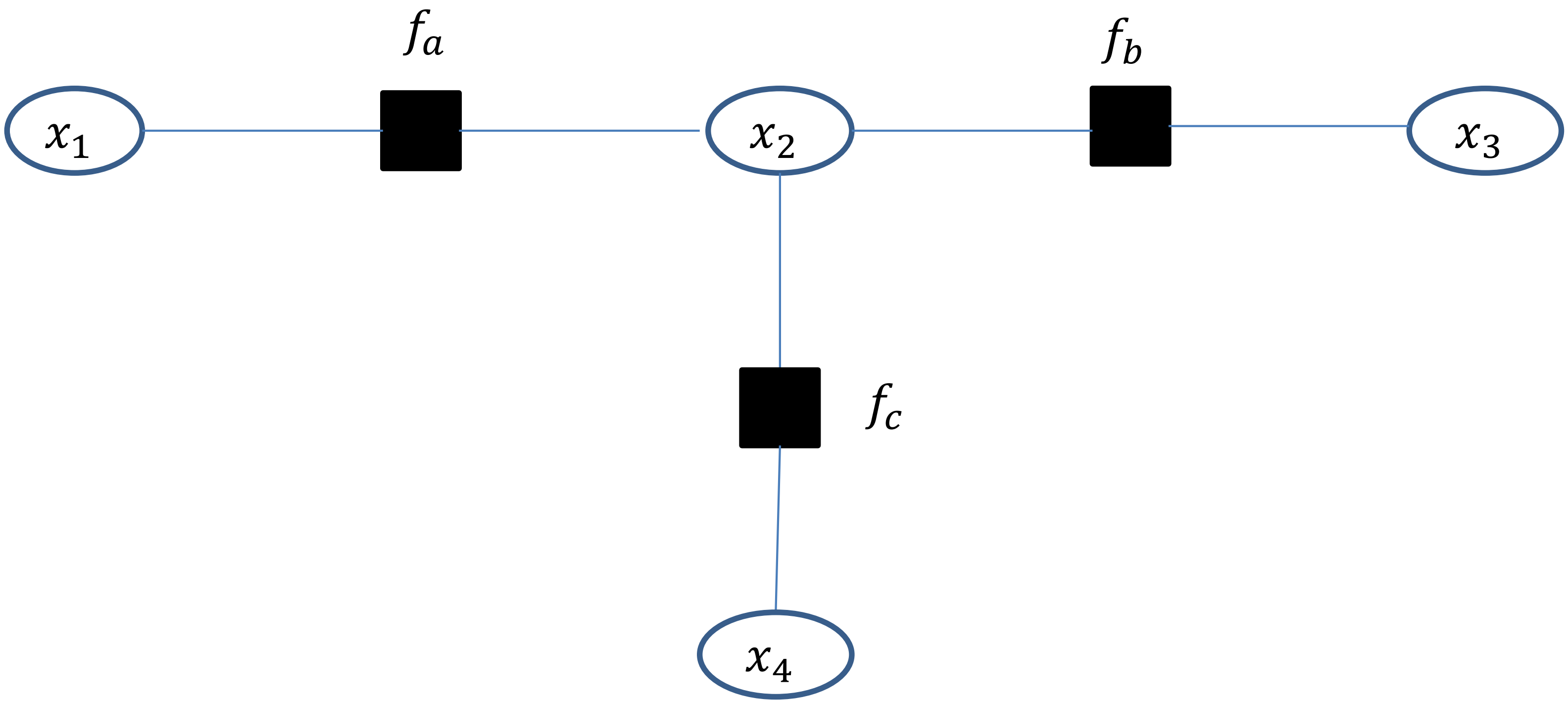
$$r_{m\to n}(x_n) = \sum_{\boldsymbol{w}_m \setminus n} f_m(\boldsymbol{w}_m) \prod_{n' \in N(m)\setminus n} \underline{q_{n'\to m}}(x_{n'})$$

sum over configuration of variables for the m^th factor, with variable n fixed

set of variables that the m^th factor depends on

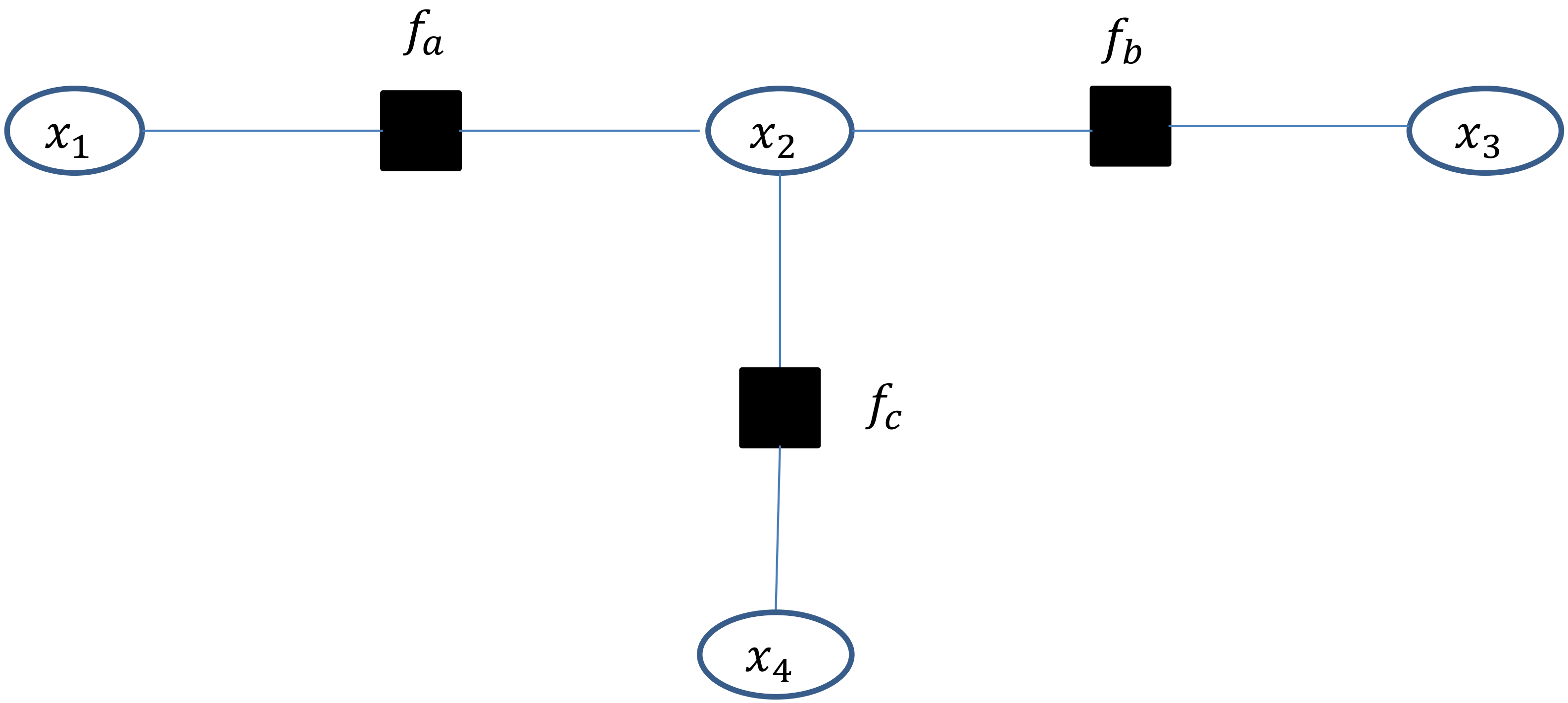


Example



Q: What are the variables?

Example

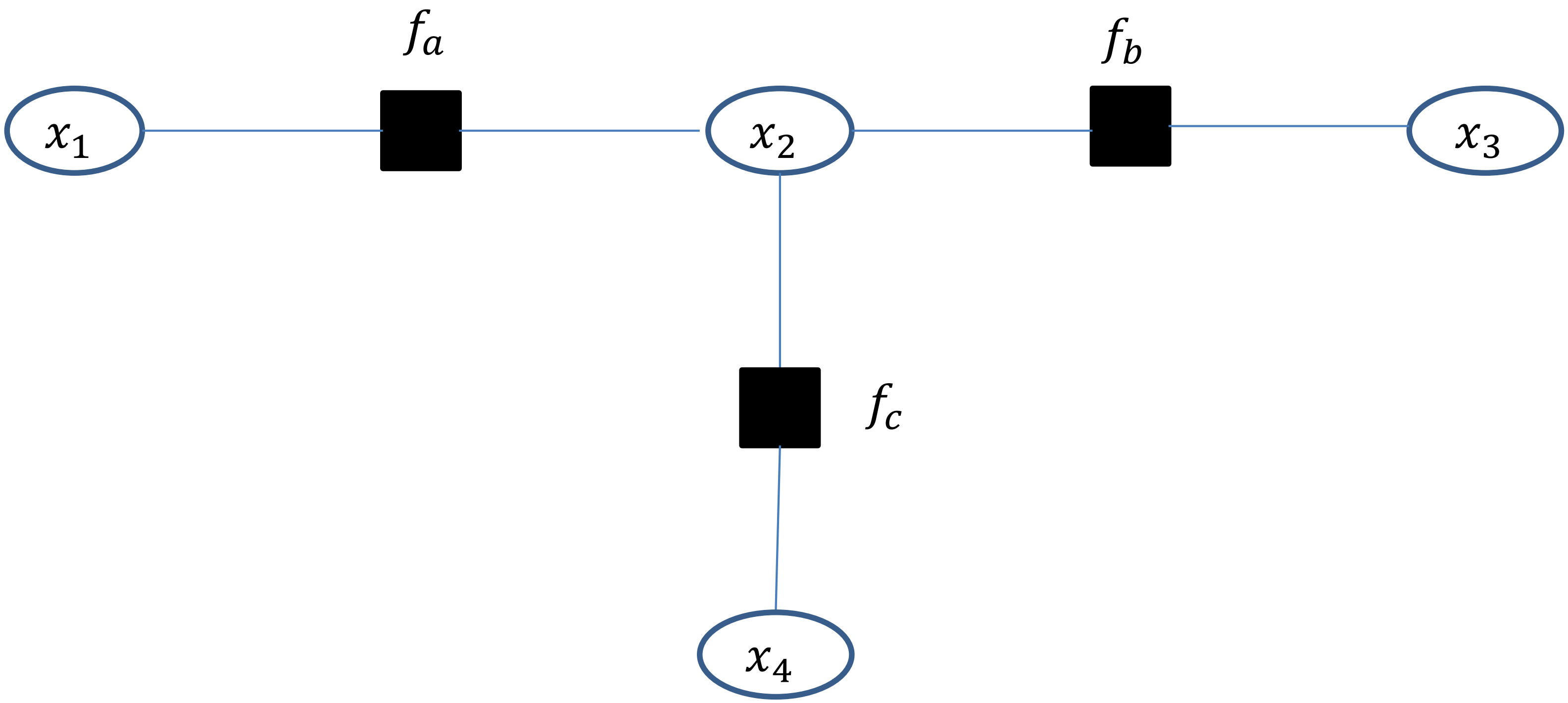


Q: What are the variables?

A: x_1, x_2, x_3, x_4

Q: What are the factors?

Example



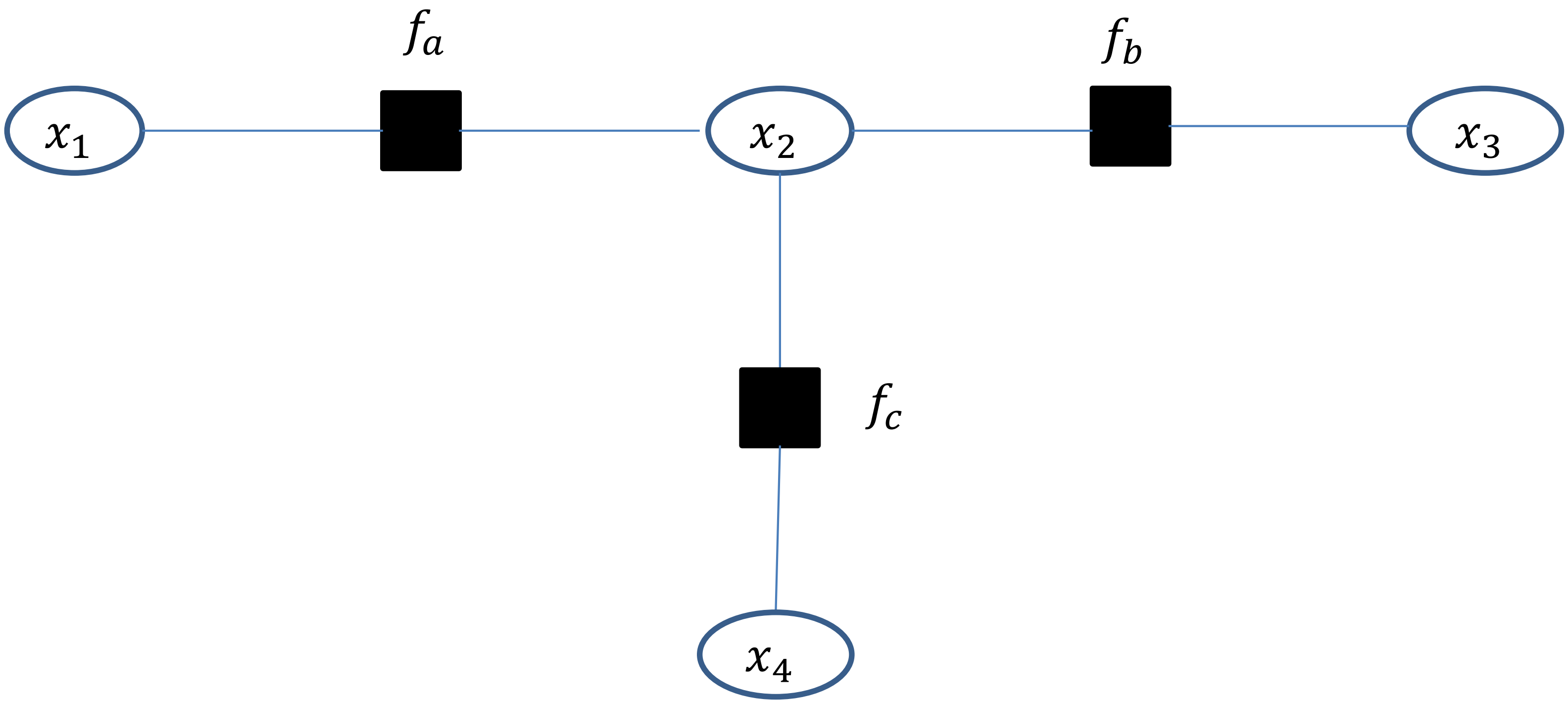
Q: What are the variables?

A: x_1, x_2, x_3, x_4

Q: What are the factors?

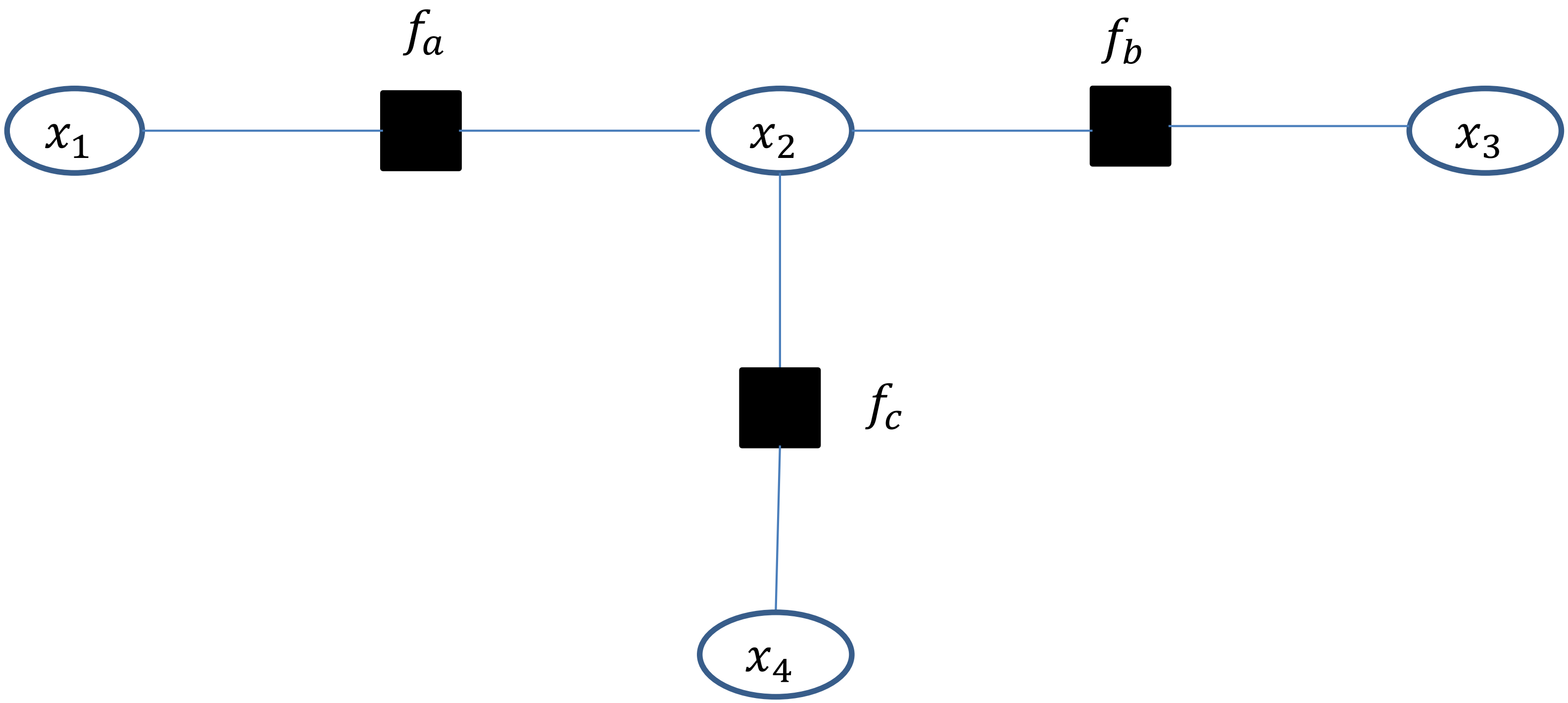
A: $f_a(x_1, x_2)$, $f_b(x_2, x_3)$, $f_c(x_2, x_4)$

Example



Q: What is the distribution we're modeling?

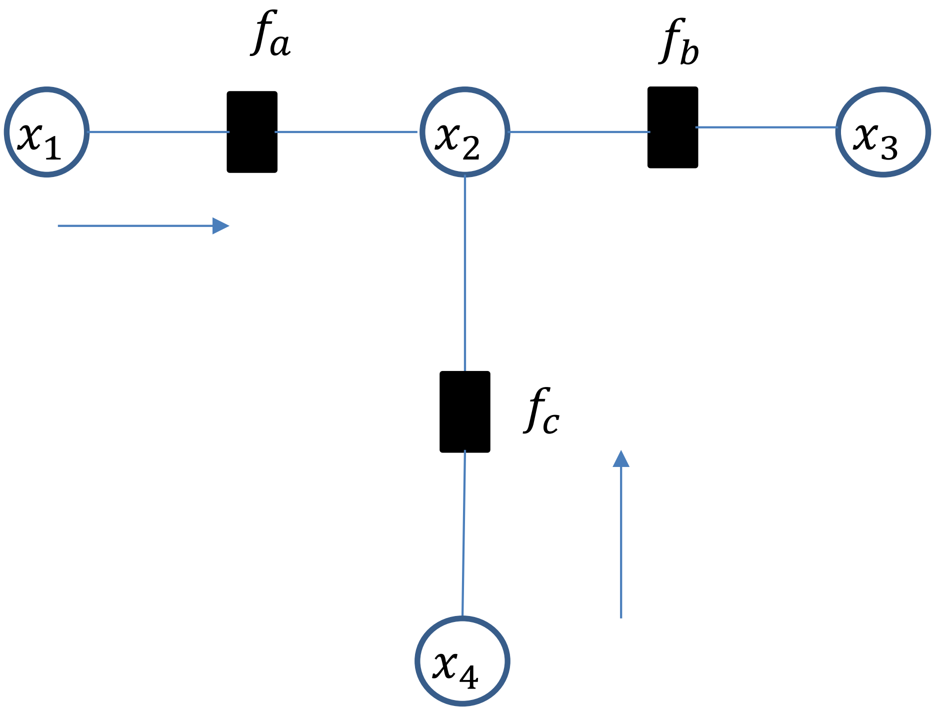
Example



Q: What is the distribution we're modeling?

A:
$$p(x_1, x_2, x_3, x_4) = f_a(x_1, x_2)\, f_b(x_2, x_3)\, f_c(x_2, x_4)$$

Example

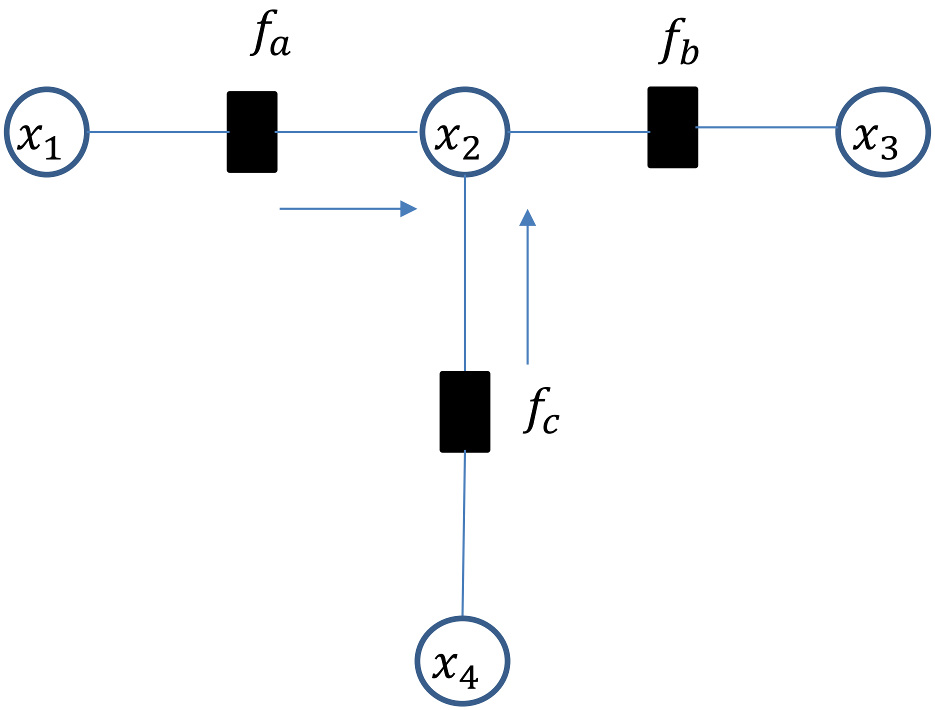


1. Select the root, or pick one if a tree (x_3)
   1. Send messages from leaves to root

$$q_{x_1 \to f_a}(x_1) = 1$$
$$q_{x_4 \to f_c}(x_4) = 1$$

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\boldsymbol{w}_m \setminus n} f_m(\boldsymbol{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Example



1. Select the root, or pick one if a tree (x_3)
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      $q_{x_1 \to f_a}(x_1) = 1$

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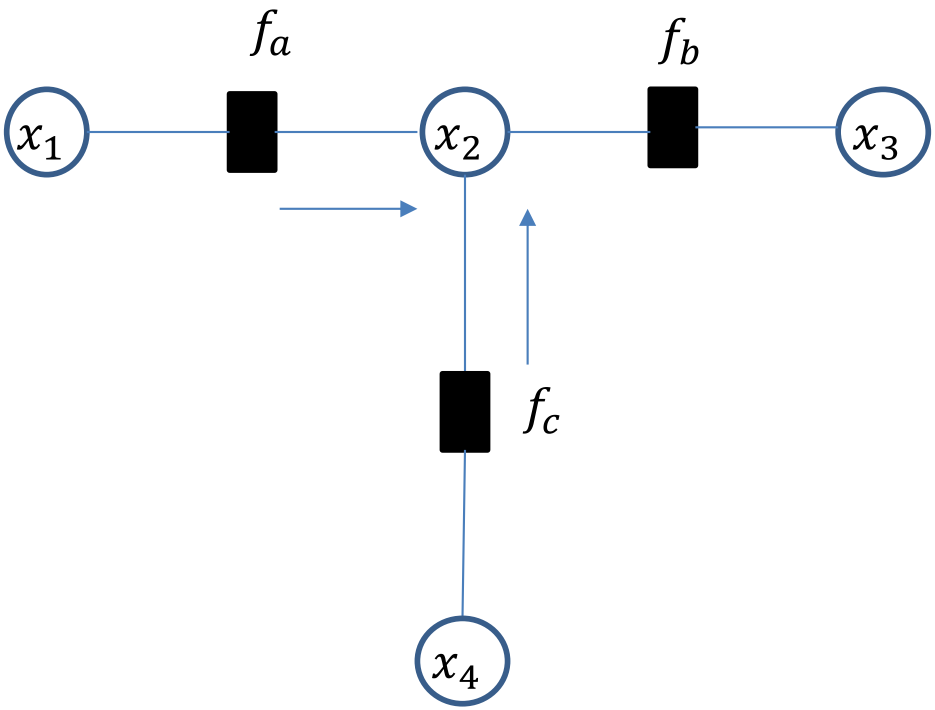
      $r_{f_a \to x_2}(x_2) = ???$

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \setminus n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Example



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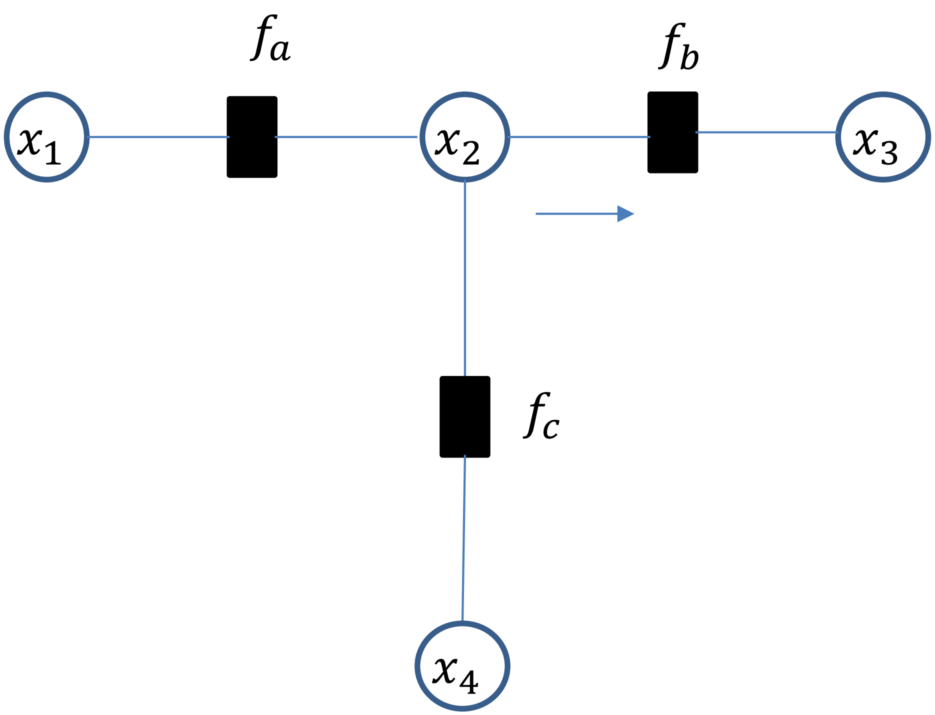
Example



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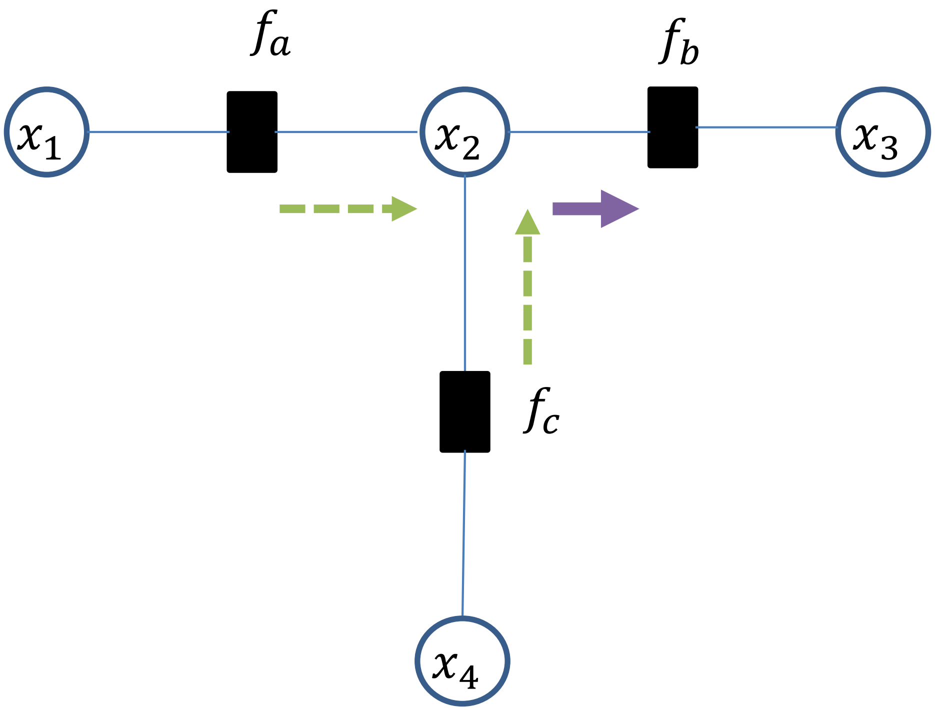
Example



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$$q_{x_2 \to f_b}(x_2) = r_{f_a \to x_2}(x_2) r_{f_c \to x_2}(x_2)$$

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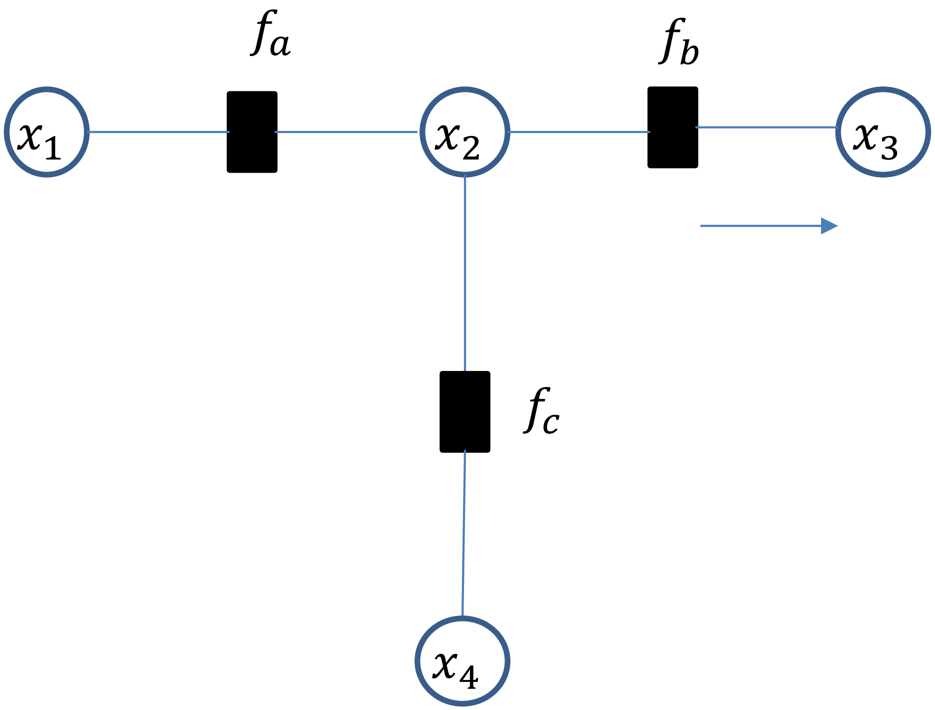
Example



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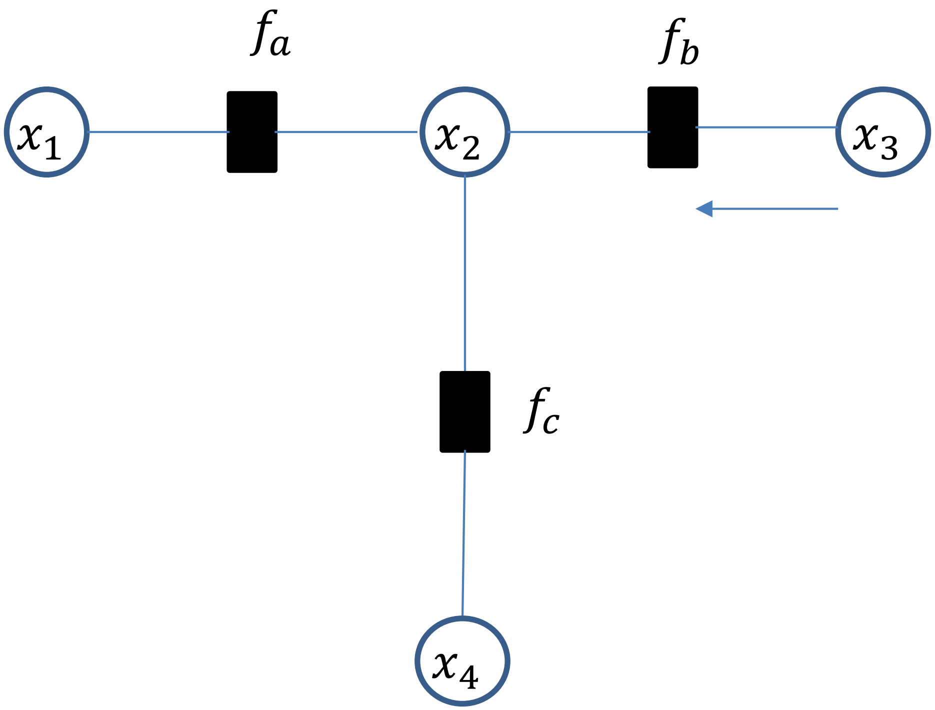
$$r_{f_b \to x_3}(x_3) = \sum_k f_b(x_2 = k, x_3)$$

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \backslash m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\boldsymbol{w}_m \backslash n} f_m(\boldsymbol{w}_m) \prod_{n' \in N(m) \backslash n} q_{n' \to m}(x_{n'})$$

Example

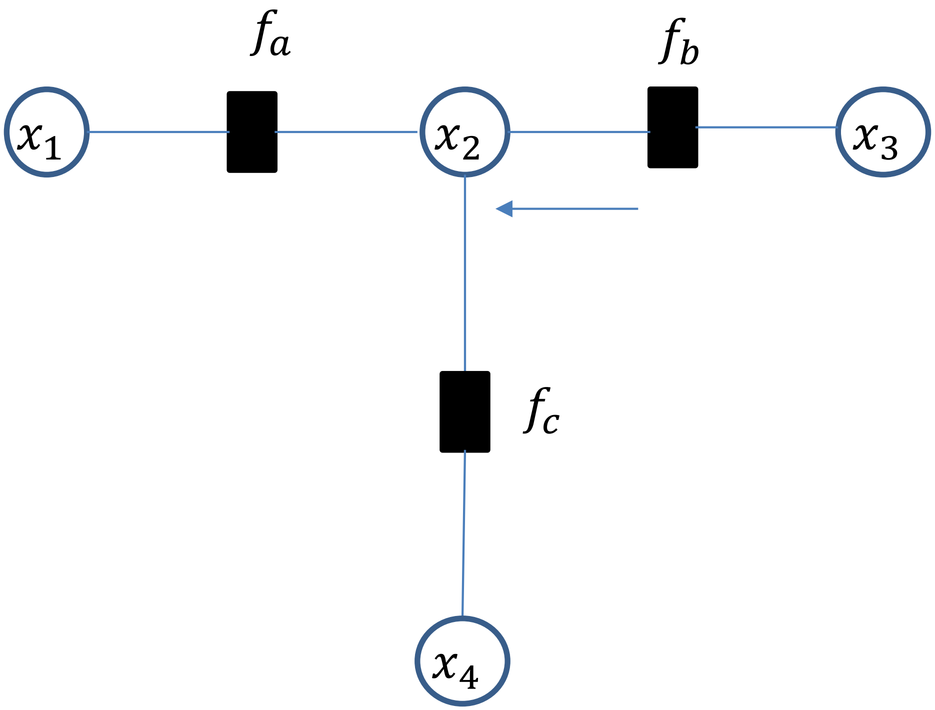


1. Select the root, or pick one if a tree (x_3)
   1. Send messages from leaves to root
   2. Send messages from root to leaves

$$q_{x_3 \to f_b}(x_3) = 1$$

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \backslash m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\boldsymbol{w}_m \backslash n} f_m(\boldsymbol{w}_m) \prod_{n' \in N(m) \backslash n} q_{n' \to m}(x_{n'})$$

Example



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   $q_{x_3 \to f_b}(x_3) = 1$

   $$r_{f_b \to x_2}(x_2) = \sum_k f_b(x_2, x_3 = k)$$

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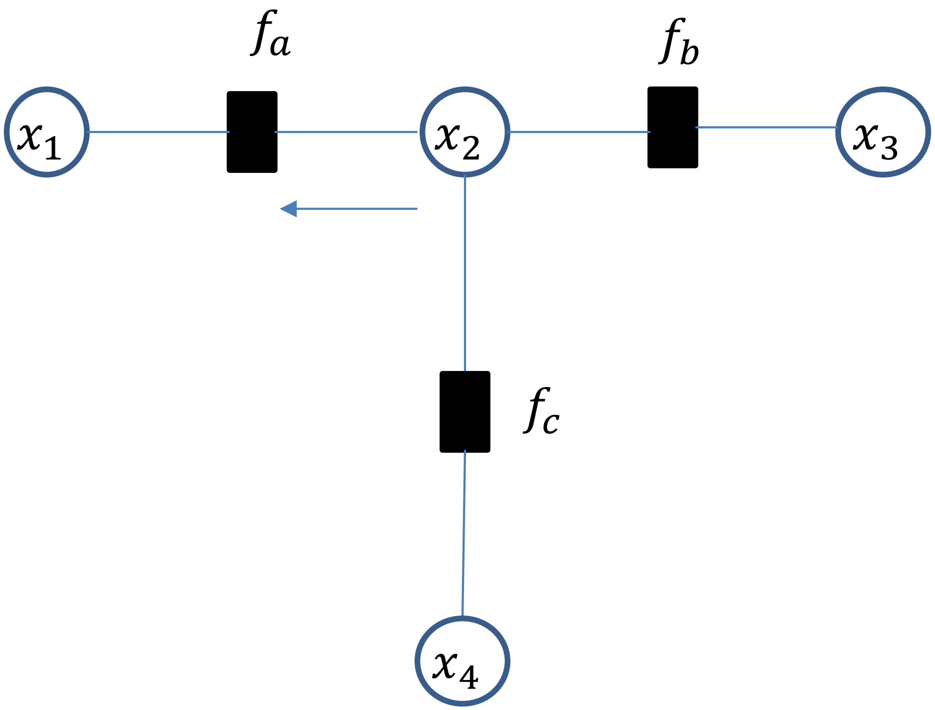
Example



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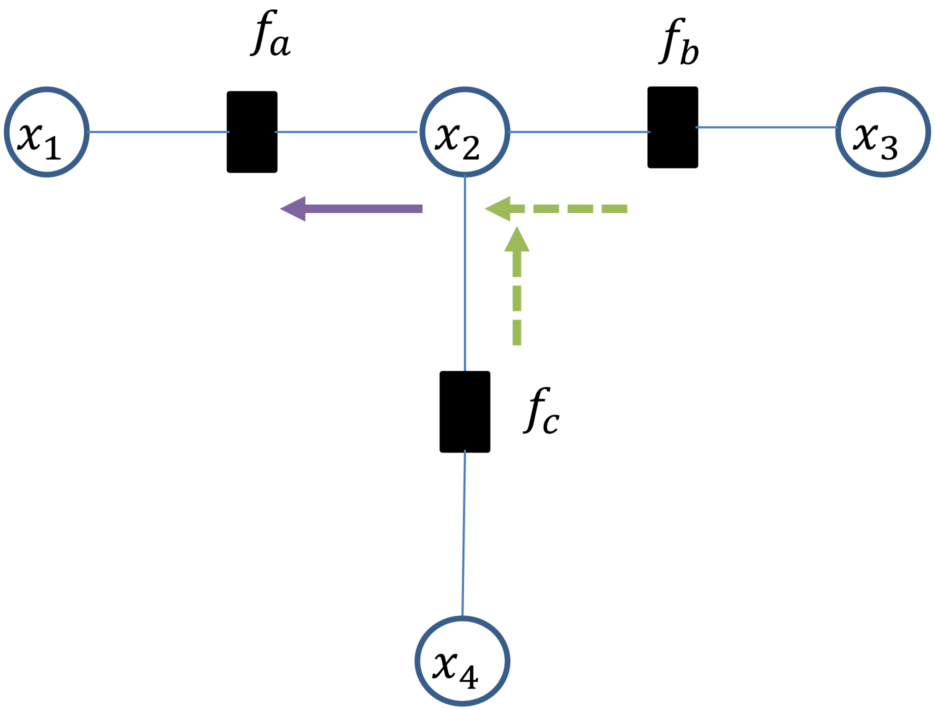
$$r_{f_b \to x_2}(x_2) = \sum_k f_b(x_2, x_3 = k)$$

$$q_{x_2 \to f_a}(x_2) = \;???$$

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \setminus n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Example



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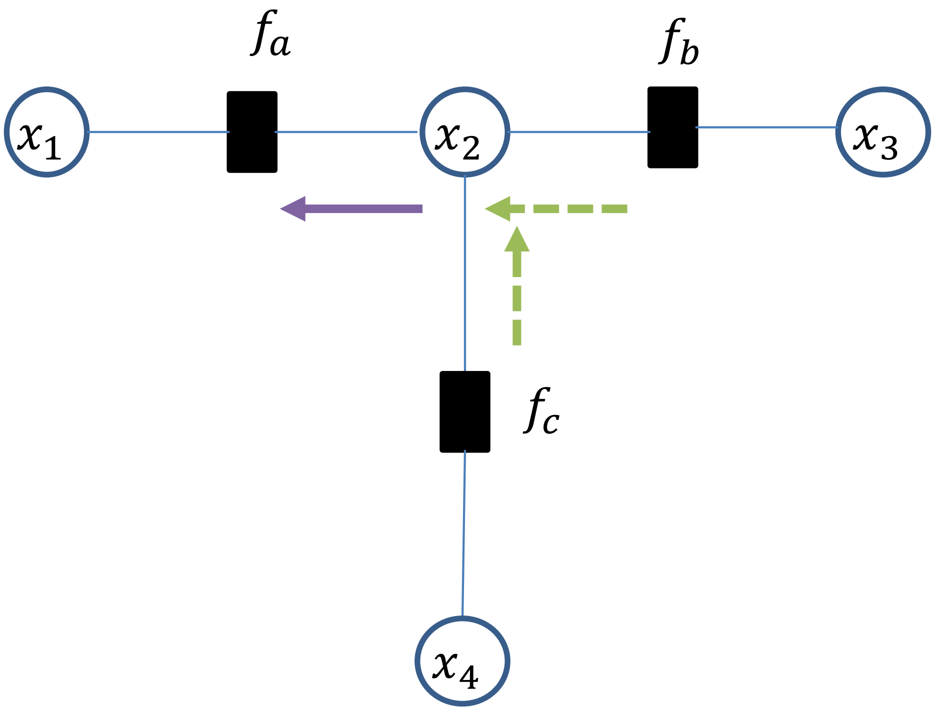
$$q_{x_2 \to f_a}(x_2) = r_{f_b \to x_2}(x_2) r_{f_c \to x_2}(x_2)$$

We just computed this

Q: Where did we compute this?

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \setminus n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Example



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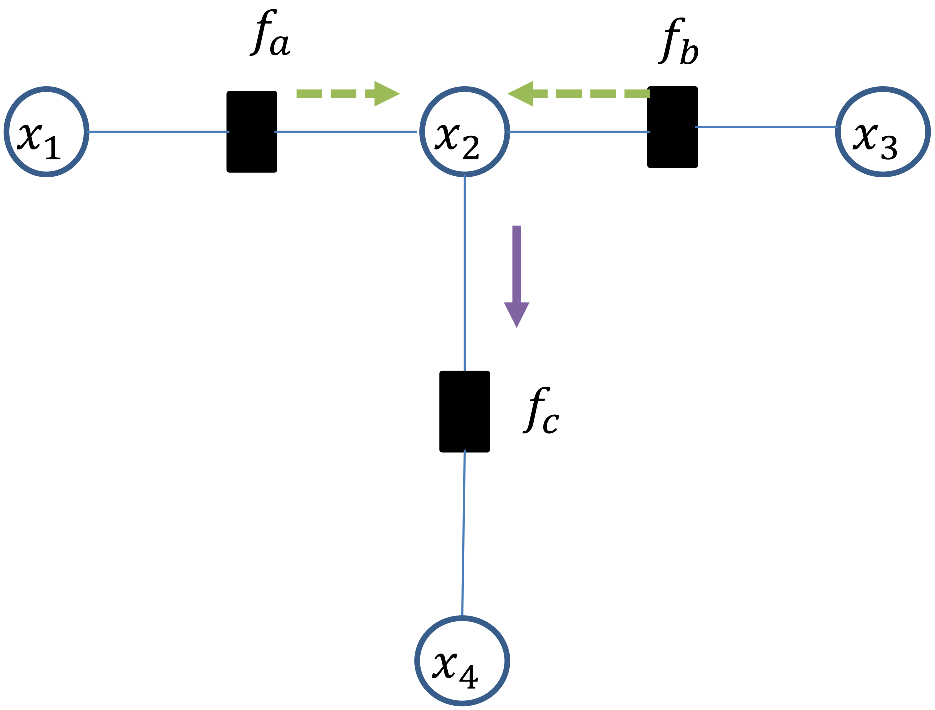
We just computed this

Q: Where did we compute this?

A: In step 1 (leaves → root)

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \setminus n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Example



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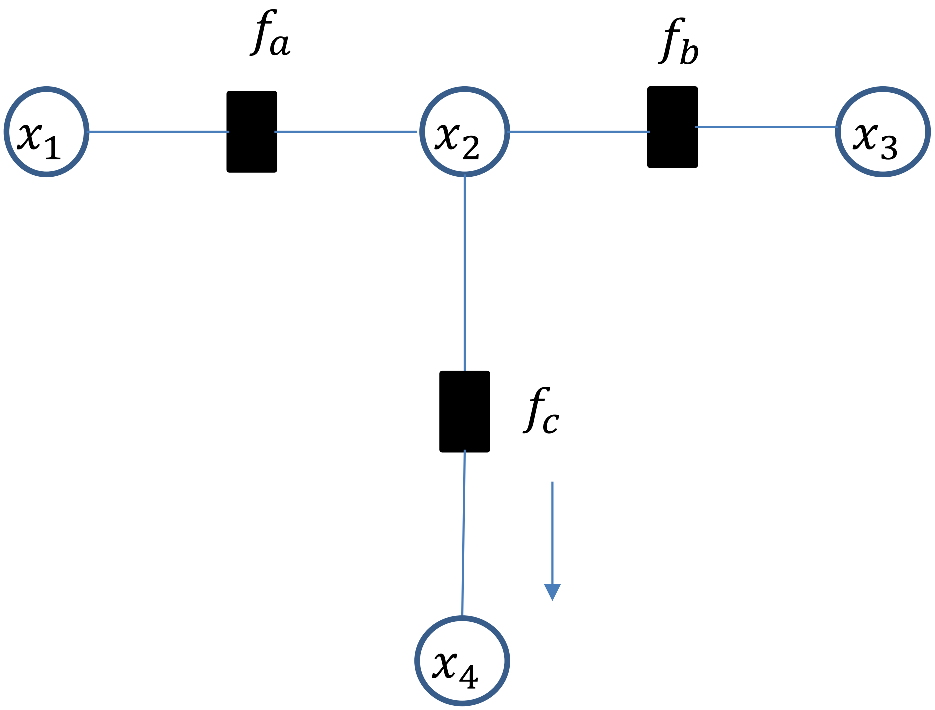
Example



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$$q_{x_2 \to f_a}(x_2) = r_{f_b \to x_2}(x_2) r_{f_c \to x_2}(x_2)$$

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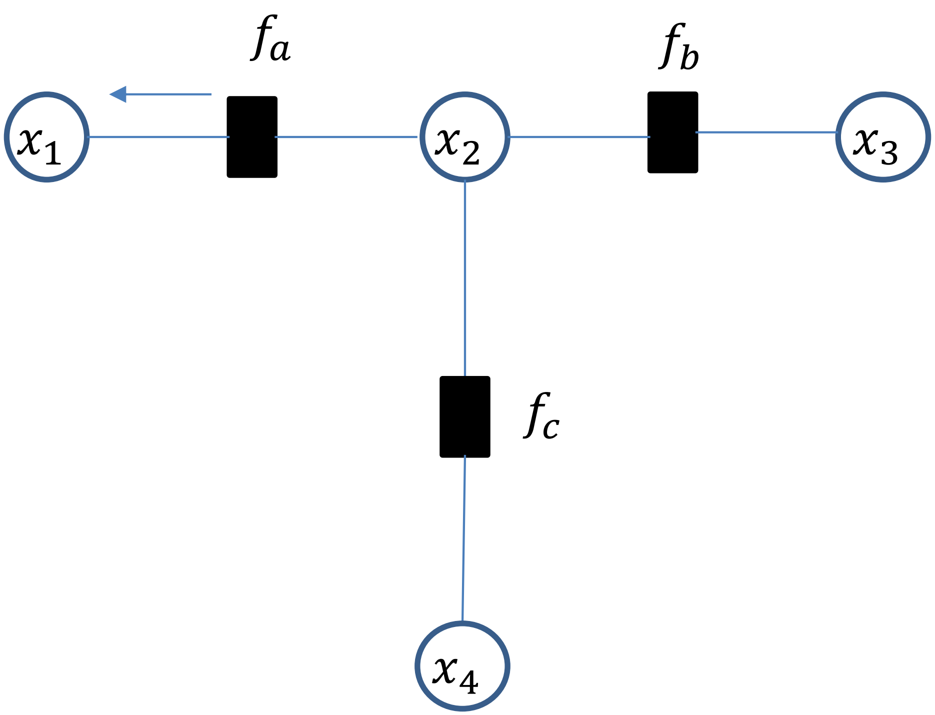
$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \setminus n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Example



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$$r_{f_c \to x_4}(x_4) = \sum_k f_c(x_2 = k, x_4)$$

$$r_{f_a \to x_1}(x_1) = \sum_k f_a(x_1, x_2 = k)$$

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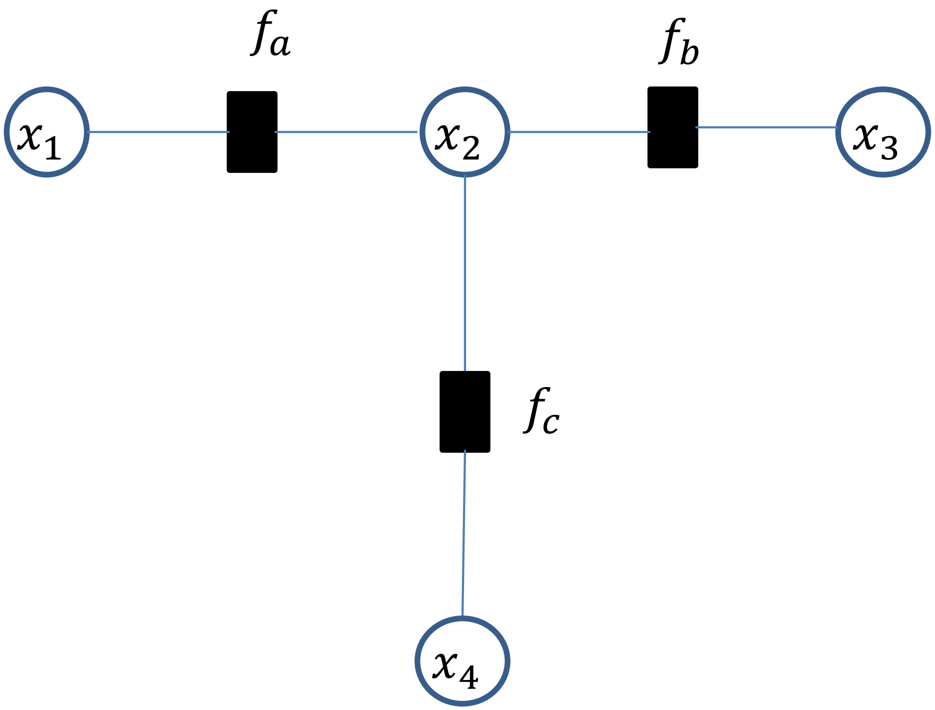
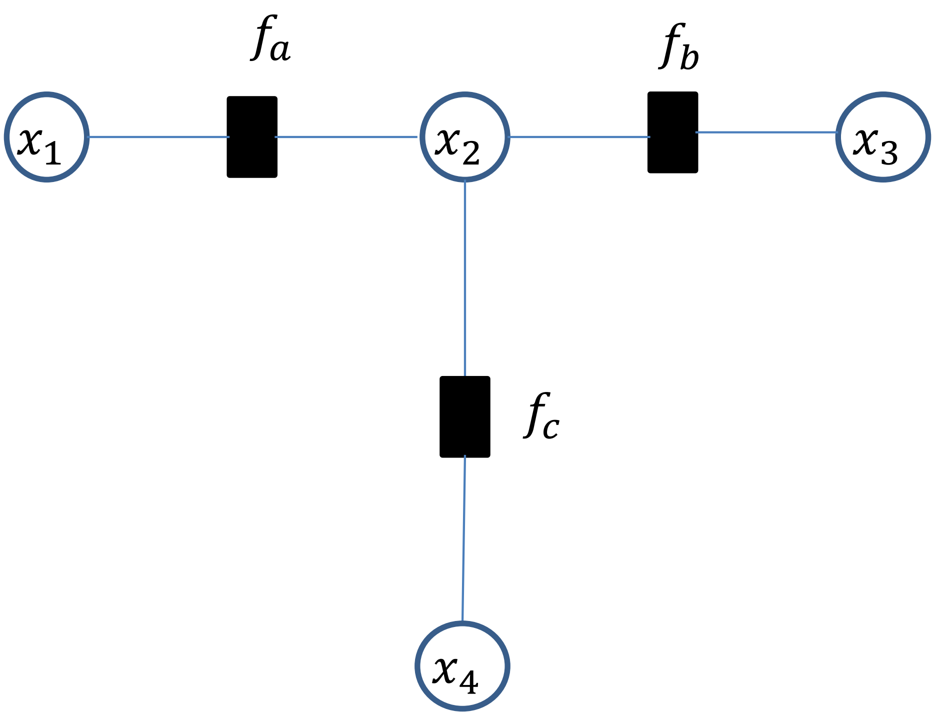


1. Select the root, or pick one if a tree (x_3)
   1. Send messages from leaves to root
   2. Send messages from root to leaves
   3. Use messages to compute marginal probabilities

$$p(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n)$$

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \setminus n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Example



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$$p(x_1) = r_{f_a \to x_1}(x_1)$$

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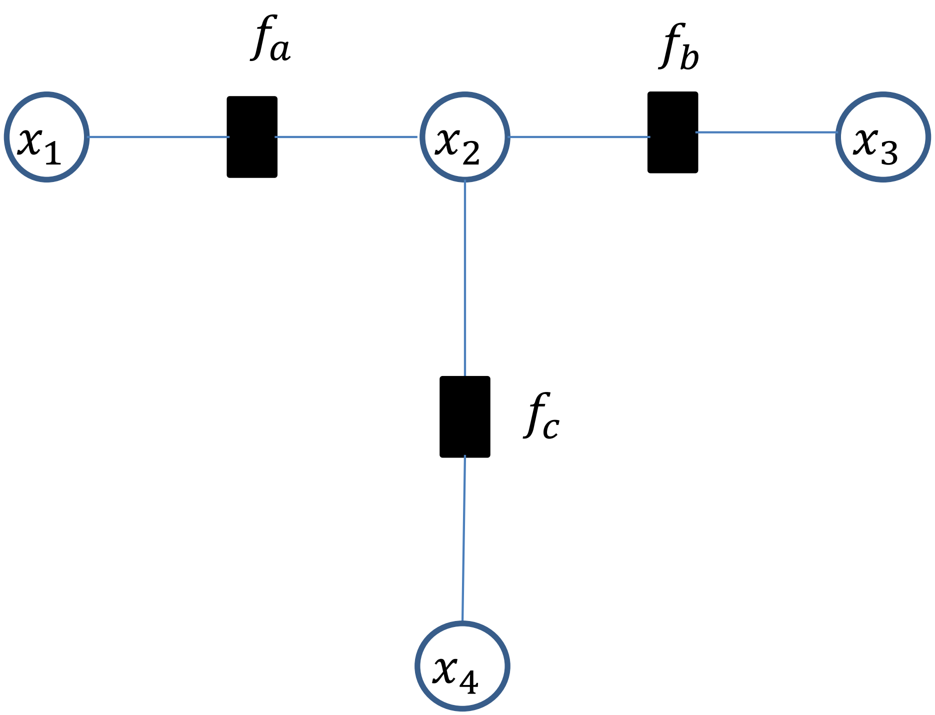
Example



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$$p(x_2) = r_{f_a \to x_2}(x_2) r_{f_b \to x_2}(x_2) r_{f_c \to x_2}(x_2)$$

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \setminus n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

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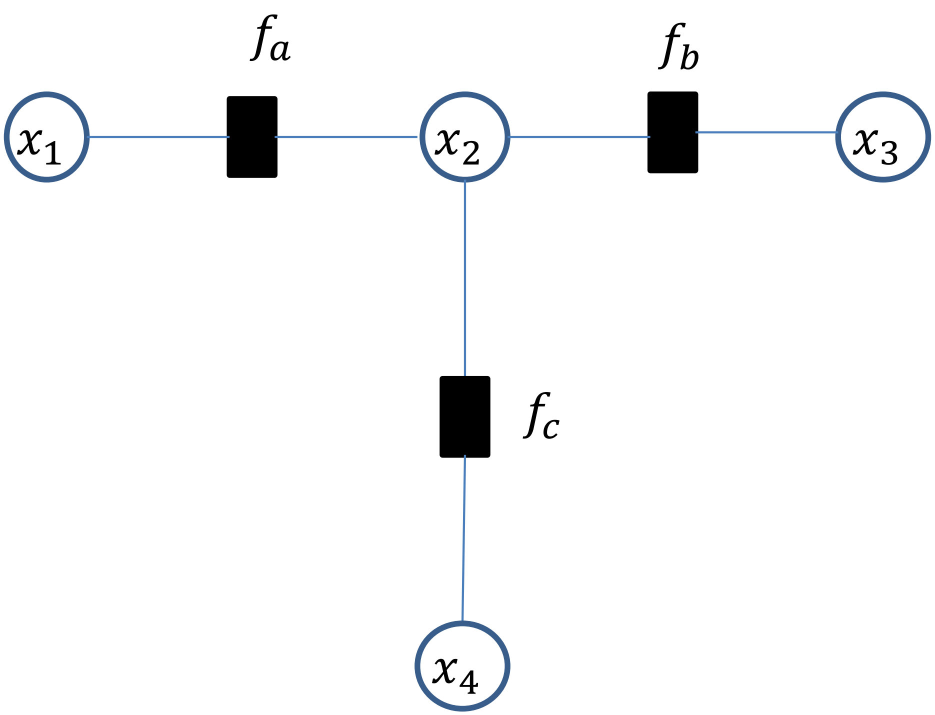

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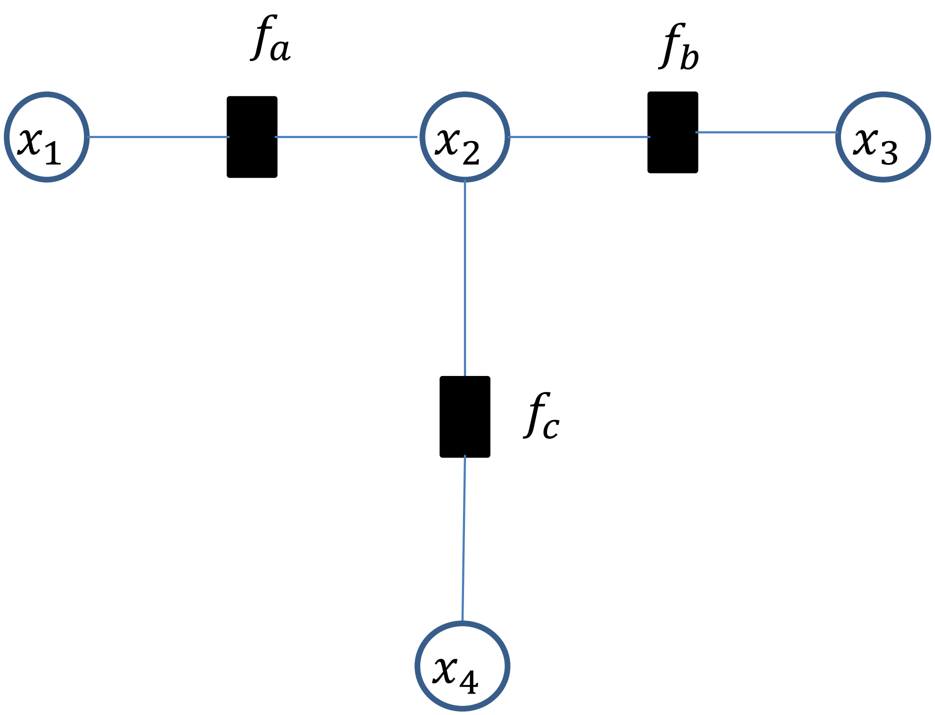
$$p(x_3) = r_{f_b \to x_3}(x_3)$$

$$p(x_4) = r_{f_c \to x_4}(x_4)$$

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \backslash m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \backslash n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \backslash n} q_{n' \to m}(x_{n'})$$

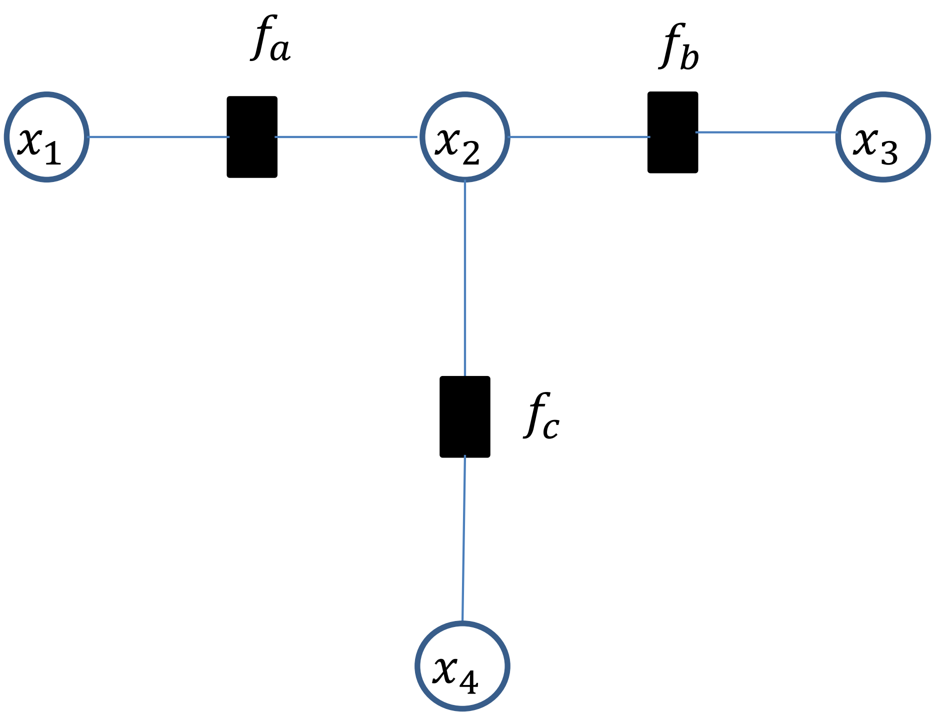
Example



1. Select the root, or pick one if a tree (x_3)
   1. Send messages from leaves to root
   2. Send messages from root to leaves
   3. Use messages to compute marginal probabilities
2. Are we done?
   1. If a tree structure, we've converged
   2.

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \backslash m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\mathbf{w}_m \backslash n} f_m(\mathbf{w}_m) \prod_{n' \in N(m) \backslash n} q_{n' \to m}(x_{n'})$$

Example



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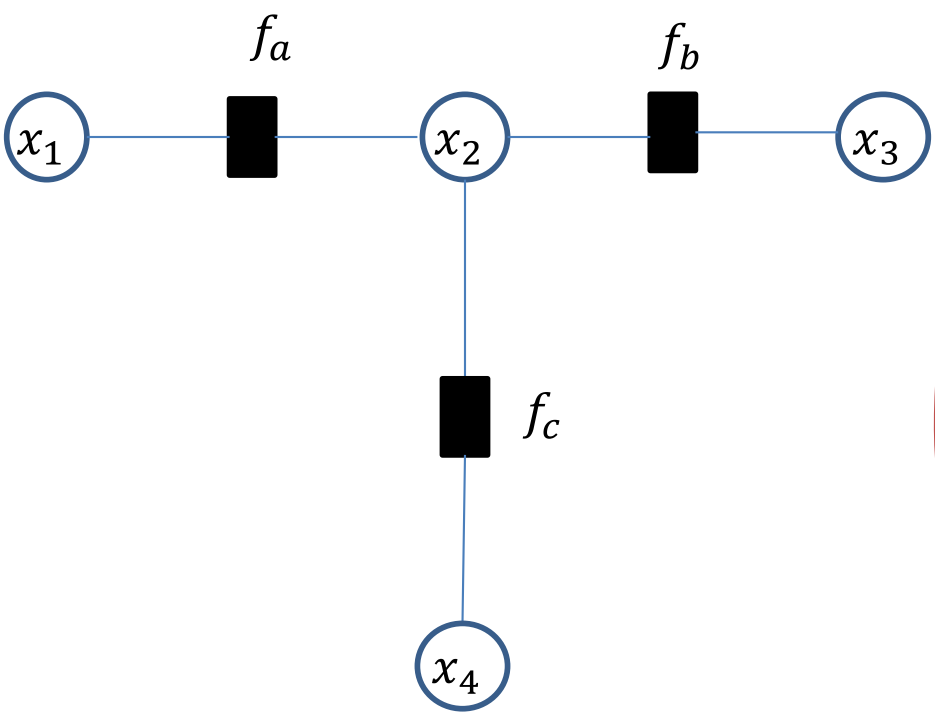
   1. If a tree structure, we've converged
   2. If not:
      1. Either accept the partially converged result, or…
      2.

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\boldsymbol{w}_m \setminus n} f_m(\boldsymbol{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Example



1. Select the root, or pick one if a tree
    1. Send messages from leaves to root
    2. Send messages from root to leaves
    3. Use messages to compute marginal probabilities
2. Are we done?
    1. If a tree structure, we've converged
    2. If not:
        1. Either accept the partially converged result, or…
        2. Go back to (1) and repeat

[Loopy BP]

$$q_{n \to m}(x_n) = \prod_{m' \in M(n) \setminus m} r_{m' \to n}(x_n) \qquad r_{m \to n}(x_n) = \sum_{\boldsymbol{w}_m \setminus n} f_m(\boldsymbol{w}_m) \prod_{n' \in N(m) \setminus n} q_{n' \to m}(x_{n'})$$

Max-Product (Max-Sum)

Problem: how to find the most likely (best) setting of latent variables

Replace sum (+) with max in factor→variable computations

$$r_{m\to n}(x_n) = \max_{\boldsymbol{w}_m \setminus n} f_m(\boldsymbol{w}_m) \prod_{n' \in N(m)\setminus n} q_{n'\to m}(x_{n'})$$

(why max-*sum*? computationally, implement with logs)

Loopy Belief Propagation

Sum-product algorithm is not exact for general graphs

Loopy Belief Propagation (Loopy BP): run sum-product algorithm *anyway* and hope for the best

Requires a **message passing schedule**

Outline

Directed Graphical Models

- Naïve Bayes

Undirected Graphical Models

- Factor Graphs
- Ising Model

Message Passing: Graphical Model Inference