Curves and Surfaces

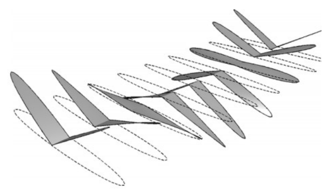
Readings: Chapter 15

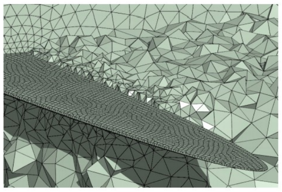
Announcement (Oct 8)

- Midterm next Wed (3/11). I will post midterm review questions online tonight.

Motivations

In many applications, we need smooth shapes.
So far we can only make things with corners:
e.g., lines, squares, triangles
Circles and ellipses only get you so far!





[Boeing]

Spline curves

Specified by a sequence of control points
Shape is guided by control points (aka control polygons)
- interpolating: Passes through points
- approximating: merely guided by points

Matrix form of spline

$$\mathbf{p}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{p}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2 + b_3(t)\mathbf{p}_3$$

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How splines depend on their controls

- Each coordinate is separate
  - The function x(t) is determined solely by the x coordinates of the control points
  - This means 1D, 2D, 3D, … curves are all really the same
- Spline curves are linear function of their controls
  - Moving a control point two inches to the right moves x(t) twice as far as moving it by one inch
  - X(t), for fixed t, is a linear combination (weighted sum) of the control points' x coordinates
  - P(t), for fixed t, is a linear combination (weighted sum) of the control points

Example: piece wise linear reconstruction of lines

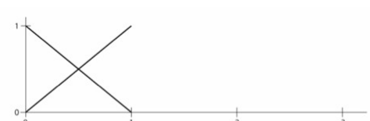
- *(See lecture notes on transforming the canonical form into the polynomial form for lines.)*
- Basis function formulation
  - Regroup expression by p rather than t

$$\begin{aligned} \mathbf{p}(t) &= (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0 \\ &= (1-t)\mathbf{p}_0 + t\mathbf{p}_1 \end{aligned}$$

  - Interpretation in matrix viewpoint

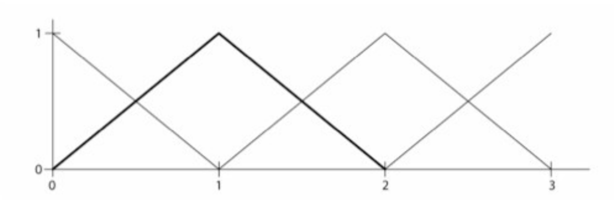
$$\mathbf{p}(t) = \left(\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

- Basis function: "function times point"
  - Contribution of each point as t changes or think about it like a reconstruction filter



Example: piece wise linear reconstruction of lines

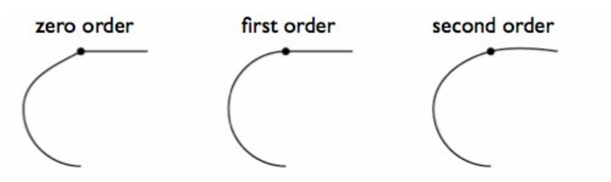
- Basis function: "function times point"
  - Basis functions: contribution of each point as t changes
  - Can think of them as blending functions glued together
  - This is like a reconstruction filter



Spline Properties

Continuity

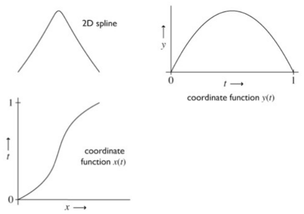
- Smoothness can be described by degree of continuity
  - Zero-order (C0): position matches from both sides: (p3 = q0 for two curves with control points of p0-3 and q0-3)
  - First-order (C1): tangent matches from both sides (p3-p2 = q1-q0)
  - Second-order (C2): curvature matches from both sides



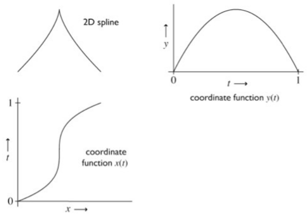
Continuity

- **Parametric continuity (C)** of spline is continuity of coordinate functions
- **Geometric continuity (G)** is continuity of the curve itself
  - Zero order (G0): positions match
  - First order (G1): two tangent vectors to be in opposite directions, but the magnitudes may be different. Or p3-p2 = k(q1-q0)
- Neither form of continuity is guaranteed by the other
  - Can be C1 but not G1 when P(t) comes to a halt (next slide)
  - Can be G1 but not C1 when the tangent vector changes length abruptly.

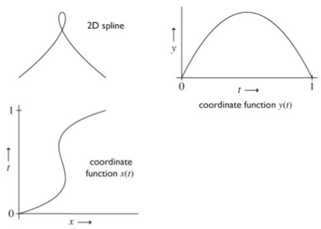
Examples: Geometric vs. parametric continuity



C1 but not G1



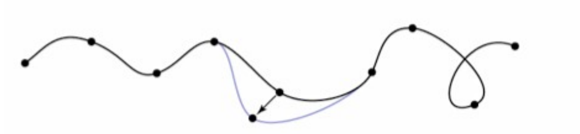
G1 but not C1



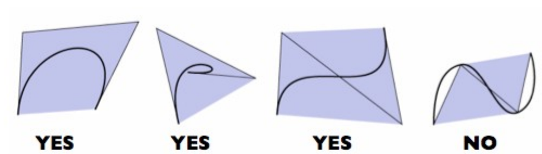
C1 and G1

Control

- Local control
  - Changing control point only affects a limited part of spline. Without this, splines are very difficult to use

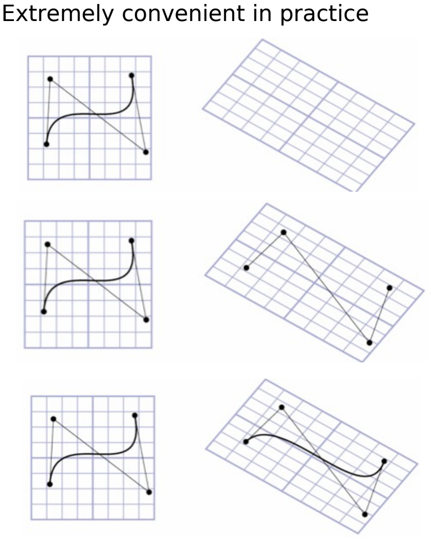


- Convex hull property
  - Convex hull = smallest convex region containing points
    - Think of a rubber band around some pins
  - Some splines stay inside convex hull of control points
    - Simplified clippings, culling, picking etc.



Affine invariance

- Transforming the control points is the same as transforming the curve
  - True for all commonly used splines
  - Extremely convenient in practice



Splines

We have discussed the matrix form of a spline

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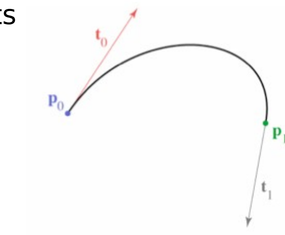
Control the basis function to form splines

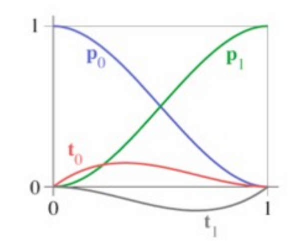
Hermite splines

- Constraints are endpoints and endpoint tangents



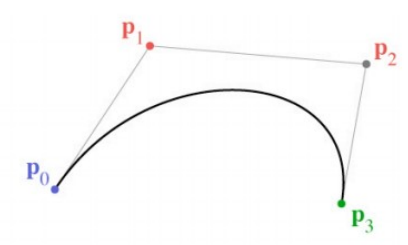
$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix}$$

- Basis function



Bezier curve

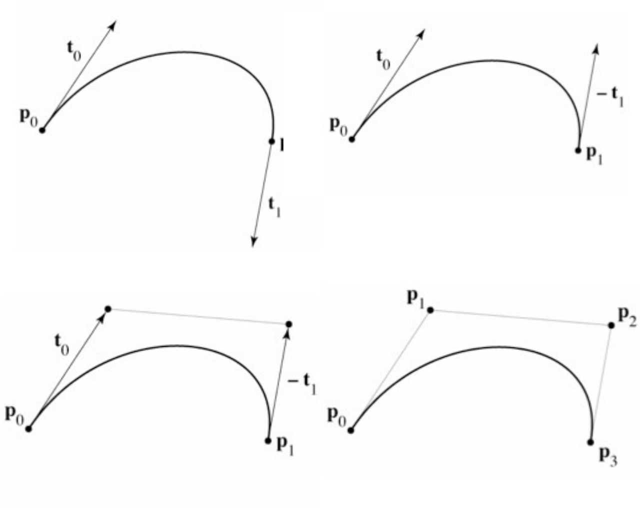
$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$



- *See lecture notes on construction from canonical form to polynomial form*

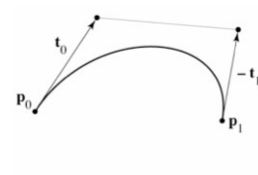
Hermite to Bezier

- Mixture of points and vectors
- Specify tangents as differences of points



Hermite to Bezier

$$\mathbf{p}_0 = \mathbf{q}_0$$
$$\mathbf{p}_1 = \mathbf{q}_3$$
$$\mathbf{v}_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$
$$\mathbf{v}_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$



$$\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\\ \mathbf{d}\end{bmatrix} = \begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{q}_0\\ \mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{bmatrix}$$

Bezier curve matrix representation

$$\mathbf{p}(t) = \begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

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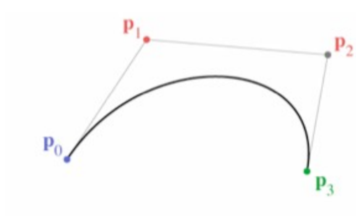
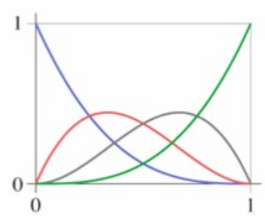




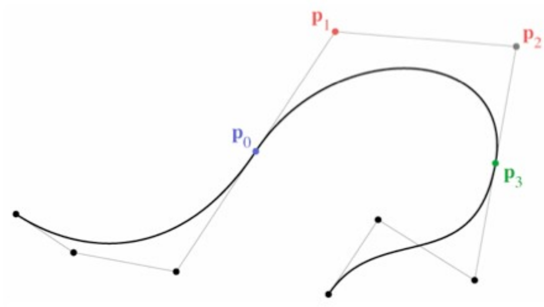
Example

- Given four points, construct the Bezier curve

$$\begin{aligned} p_0 &= (0.9, 1)\\ p_1 &= (0.9, 0)\\ p_2 &= (0, 0.5)\\ p_3 &= (1, 0.5)\end{aligned}$$

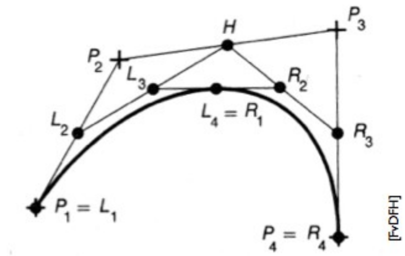
Chaining spline segments

- Bezier curves are convenient because their controls are all points and they have nice properties
  - And they interpolate every 4th point
- No continuity built in
  - Achieve C1 using colinear control points



Subdivision

- A Bezier spline segment can be split into a two segment curve



Bezier surfaces

- Can be directly derived from the 2D form (*equations are in lecture notes*)