CMSC 341

Math Review

Exponents

- Identities

$$(X^A)^B = X^{AB}$$

$$X^A * X^B = X^{A+B}$$

$$X^A / X^B = X^{A-B}$$

$$X^A + X^B \neq X^{A+B}$$

Logarithms

- Definition: $N = \log_A X$ if and only if $A^N = X$
- In this course and text, all logarithms are base 2

unless otherwise noted

- Identities

$$\log A^k = k \log A$$

$$\log AB = \log A + \log B; \ \log(A/B) = \log A - \log B$$

Mathematical series

Geometric series:

$$2^0 + 2^1 \cdots + 2^N = \sum_{i=0}^{N} 2^i = 2^{N+1} - 1$$

An observation : $2^{N+1} > 2^0 + 2^1 \cdots + 2^N$

$$\sum_{i=M}^{N} A^i = \begin{cases} \dfrac{A^{N+1} - A^M}{A - 1} & \text{if } A > 1; \\[2ex] \dfrac{A^M - A^{N+1}}{1 - A} & \text{if } A < 1); \end{cases}$$

Mathematical series (cont.)

Infinite series. Ex. $\sum_{i=0}^{\infty} A^i = \lim_{i\to\infty} \frac{1-A^{i+1}}{1-A} = \frac{1}{1-A} \quad (A<1)$

$$S = 1 + A + A^2 + A^3 + A^4 \cdots$$
$$AS = A + A^2 + A^3 + A^4 + A^5 \cdots$$
$$S - AS = 1, \text{ then } S = 1/(1-A)$$

Other series:

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2} \approx \frac{N^2}{2}; \quad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6} \approx \frac{N^3}{3}$$

Proof by induction

Three steps: to prove a theorem F(N) for any positive integer N

Step 1: ***Base case***: prove F(1) is true

there may be different base cases (or more than one base)

Step 2: ***Hypothesis***: assume F(k) is true for any k >= 1

(it is an assumption, don't try to prove it)

Step 3: ***Inductive proof***:

prove that **if** F(k) is true (assumption) **then** F(k+1) is true

F(1) from base case

F(2) from F(1) and inductive proof

F(3) from F(2) and inductive proof

…

F(k+1) from F(k) and inductive proof

Proof by induction

Ex., show that $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

Base case: N = 1:

LHS: $\sum_{i=1}^{1} i^2 = 1.$

RHS:

$1 \cdot (1+1) \cdot (2 \cdot 1 + 1)/6 = 1$

The theorem holds.

Hypothesis : assume

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

holds for any $n \geq 1$.

Inductive proof

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{(n+1)(n(2n+1) + 6(n+1))}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

exercise

Fibonacci numbers:

- 0, 1, 1, 2, 3, 5, 8, 13,…
- Formal definition:

$F(0) = 0$; $F(1) = 1$; $F(n) = F(n-1) + F(n-2)$ for $n > 1$.

Show that $\sum_{i=1}^{N} F(i) = F(N+2) - 1$

(We showed this in class.)