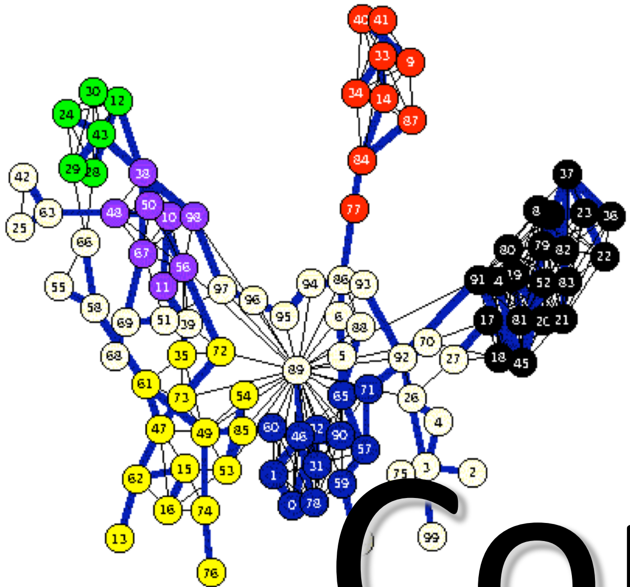


Constraint Satisfaction

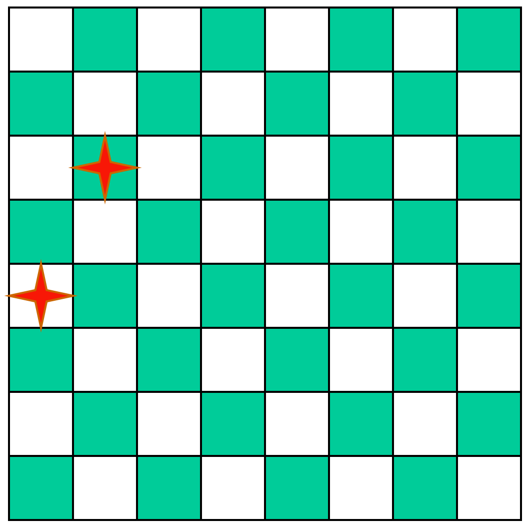
Russell & Norvig Ch. 6

Overview

- Constraint satisfaction is a powerful problem-solving paradigm
  - Problem: set of variables to which we must assign values satisfying problem-specific constraints
  - Constraint programming, constraint satisfaction problems (CSPs), constraint logic programming…
- Algorithms for CSPs
  - Backtracking (systematic search)
  - Constraint propagation (k-consistency)
  - Variable and value ordering heuristics
  - Backjumping and dependency-directed backtracking

Motivating example: 8 Queens

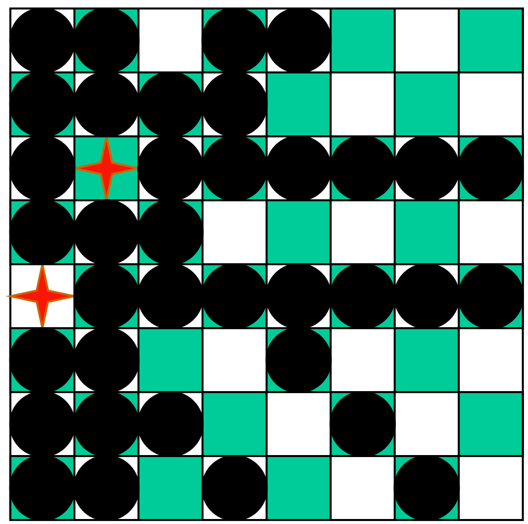
Place 8 queens on a chess board such
That none is attacking another.



Generate-and-test, with no
redundancies → "only" 8^8 combinations

8**8 is 16,777,216

Motivating example: 8-Queens



After placing these two queens, it's trivial to mark the squares we can no longer use

What more do we need for 8 queens?

- Not just a successor function and goal test
- But also
  - a means to propagate constraints imposed by one queen on others
  - an early failure test

→ Explicit representation of constraints and constraint manipulation algorithms

Informal definition of CSP

- CSP (Constraint Satisfaction Problem), given

  (1) finite set of variables

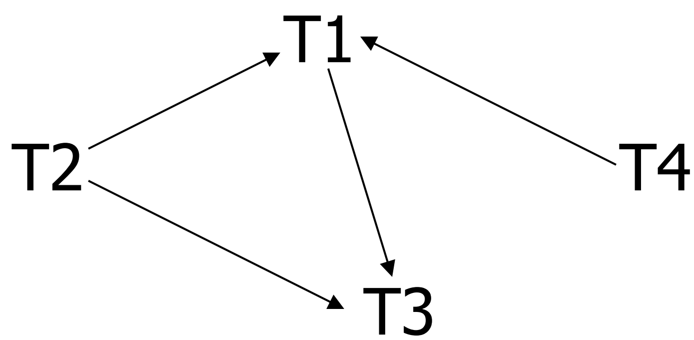
  (2) each with domain of possible values (often finite)

  (3) set of constraints limiting values variables can take
- Solution: assignment of a value to each variable such that all constraints are satisfied
- Possible tasks: decide if solution exists, find a solution, find all solutions, find *best solution* according to some metric (objective function)

Example: 8-Queens Problem

- Eight variables Qi, i = 1..8 where Qi is the row number of queen in column i
- Domain for each variable {1,2,…,8}
- Constraints are of the forms:
  - No queens on same row
    Qi = k ➔ Qj $\neq$ k for j = 1..8, j$\neq$i
  - No queens on same diagonal
    Qi=rowi, Qj=rowj ➔ |i-j| $\neq$ |rowi-rowj| for j = 1..8, j$\neq$i

Example: Task Scheduling

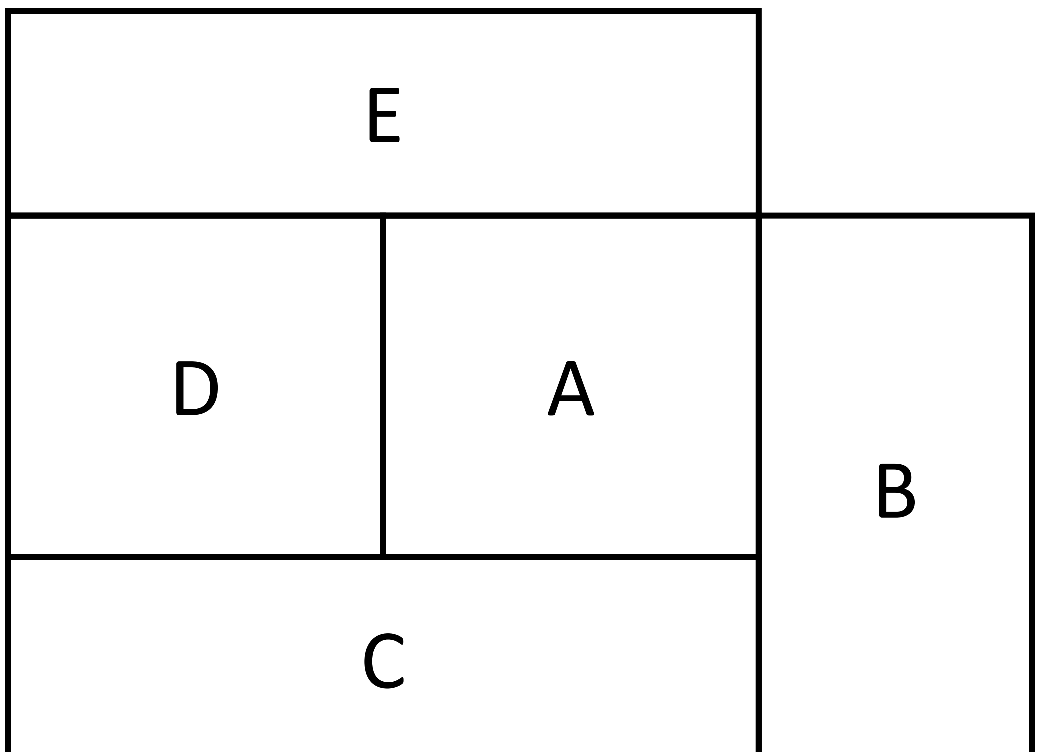


Examples of scheduling constraints:

- T1 must be done during T3
- T2 must be achieved before T1 starts
- T2 must overlap with T3
- T4 must start after T1 is complete

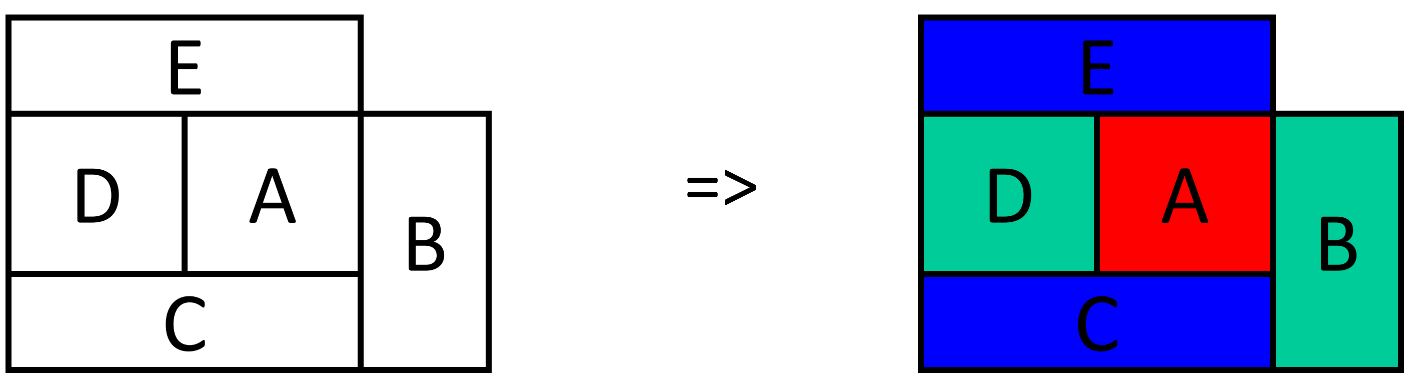
Example: Map coloring

Color this map using three colors (red, green, blue) such that no two adjacent regions have the same color



Map coloring

- Variables: A, B, C, D, E all of domain RGB
- Domains: RGB = {red, green, blue}
- Constraints: A≠B, A≠C, A≠E, A≠D, B≠C, C≠D, D≠E
- A solution: A=red, B=green, C=blue, D=green, E=blue



Brute Force methods

- Finding a solution by a brute force search is easy
  - Generate and test is a *weak method*
  - Just generate potential combinations and test each
- Potentially very inefficient
  - With n variables where each can have one of 3 values, there are 3^n possible solutions to check
- There are ~190 countries in the world, which we can color using four colors
- 4^{190} is a big number!

```
solve(A,B,C,D,E) :-
  color(A),
  color(B),
  color(C),
  color(D),
  color(E),
  not(A=B),
  not(A=B),
  not(B=C),
  not(A=C),
  not(C=D),
  not(A=E),
  not(C=D).

color(red).
color(green).
color(blue).
```

generate: color(A) … color(E)

test: not(A=B) … not(C=D)

4**190 is 2462625387274654950767440006258975862817483704404090416746768337765357610718575663213391640930307227550414249394176L

Example: SATisfiability

- Given a set of logic propositions containing variables, find an assignment of the variables to {false, true} that satisfies them
- For example, the two clauses:
  - $(A \vee B \vee \neg C) \wedge (\neg A \vee D)$
  - equivalent to $(C \rightarrow A) \vee (B \wedge D \rightarrow A)$

  are satisfied by

  A = false, B = true, C = false, D = false
- Satisfiability is known to be NP-complete, so in worst case, solving CSP problems requires exponential time

Real-world problems

CSPs are a good match for many practical problems that arise in the real world

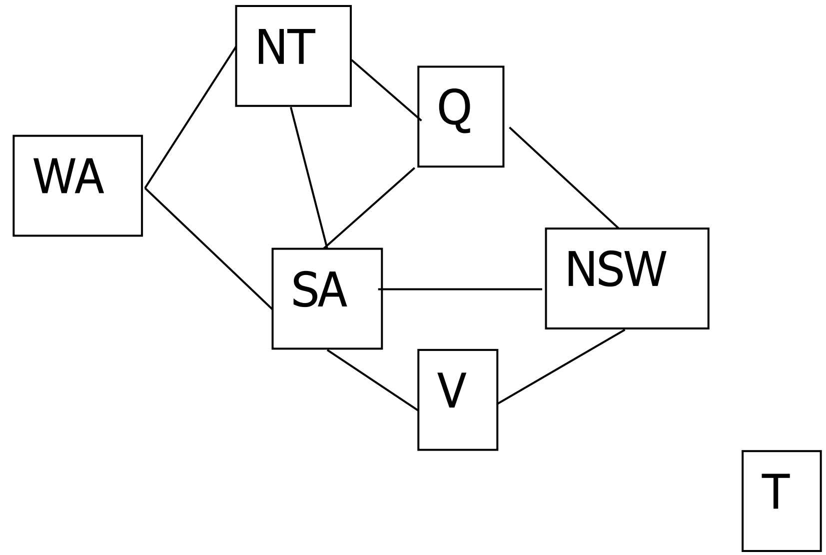
- Scheduling
- Temporal reasoning
- Building design
- Planning
- Optimization/satisfaction
- Vision
- Graph layout
- Network management
- Natural language processing
- Molecular biology / genomics
- VLSI design

Definition of a constraint network (CN)

A constraint network (CN) consists of

- Set of variables $X = \{x_1, x_2, \ldots x_n\}$
  - with associate domains $\{d_1, d_2, \ldots d_n\}$
  - domains are typically finite
- Set of constraints $\{c_1, c_2 \ldots c_m\}$ where
  - each defines a predicate that is a relation over a particular subset of variables (X)
  - e.g., C_i involves variables $\{X_{i1}, X_{i2}, \ldots X_{ik}\}$ and defines the relation $R_i \subseteq D_{i1} \times D_{i2} \times \ldots D_{ik}$

Running example: coloring Australia

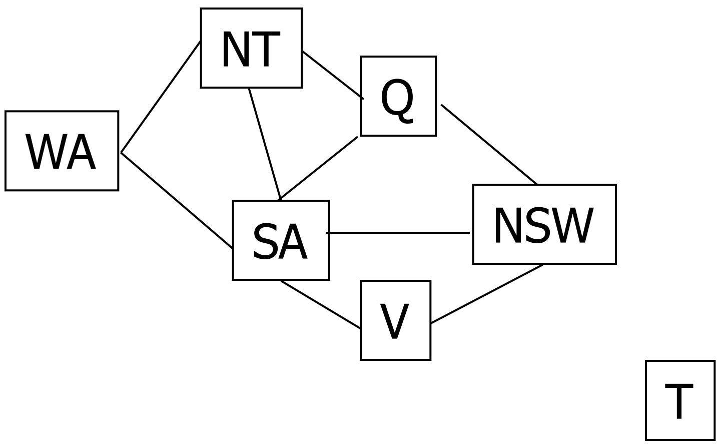


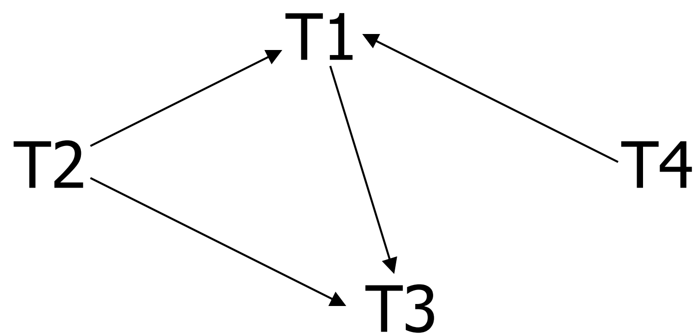
- Seven variables: {WA, NT, SA, Q, NSW, V, T}
- Each variable has same domain: {red, green, blue}
- No two adjacent variables can have same value:

WA≠NT, WA≠SA, NT≠SA, NT≠Q, SA≠Q, SA≠NSW, SA≠V,Q≠NSW, NSW≠V

Unary & binary constraints most common

Binary constraints





- Two variables are adjacent or neighbors if connected by an edge or an arc
- Possible to rewrite problems with higher-order constraints as ones with just binary constraints

Formal definition of a CN

- Instantiations
  - An instantiation of a subset of variables S is an assignment of a value (in its domain) to each variable in S
  - An instantiation is legal iff it violates no constraints
- A solution is a legal instantiation of all variables in the network

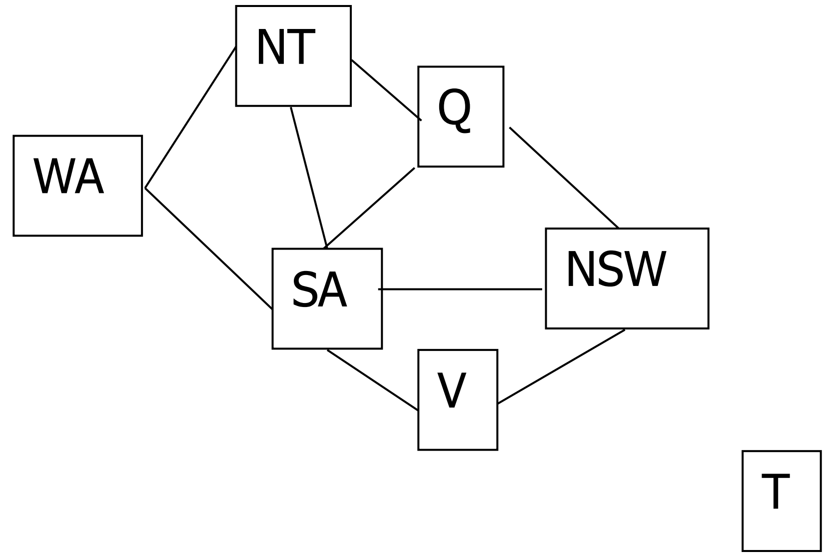
Typical tasks for CSP

- Solution related tasks:
  - Does a solution exist?
  - Find one solution
  - Find all solutions
  - Given a metric on solutions, find best one
  - Given a partial instantiation, do any of above
- Transform the CN into an equivalent CN that is easier to solve

Binary CSP

- A binary CSP is a CSP where all constraints are binary or unary
- Any non-binary CSP can be converted into a binary CSP by introducing additional variables
- A binary CSP can be represented as a constraint graph, with a node for each variable and an arc between two nodes iff there's a constraint involving them
  - Unary constraints appear as self-referential arcs

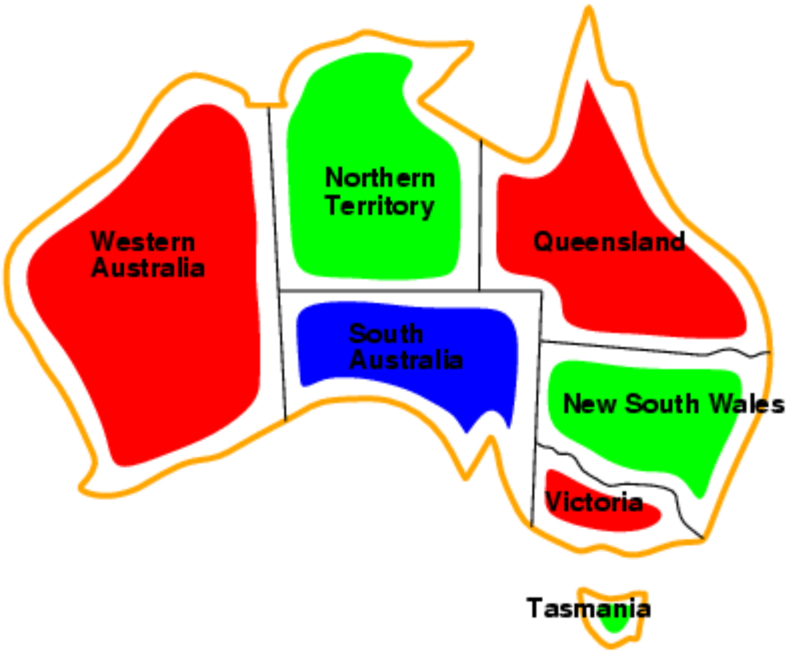
Running example: coloring Australia

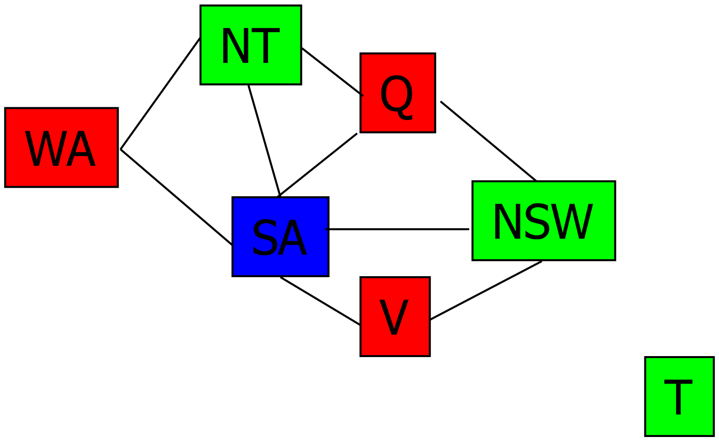


- Seven variables: {WA, NT, SA, Q, NSW, V, T}
- Each variable has same domain: {red, green, blue}
- No two adjacent variables can have same value:

WA≠NT, WA≠SA, NT≠SA, NT≠Q, SA≠Q, SA≠NSW, SA≠V,Q≠NSW, NSW≠V

A running example: coloring Australia



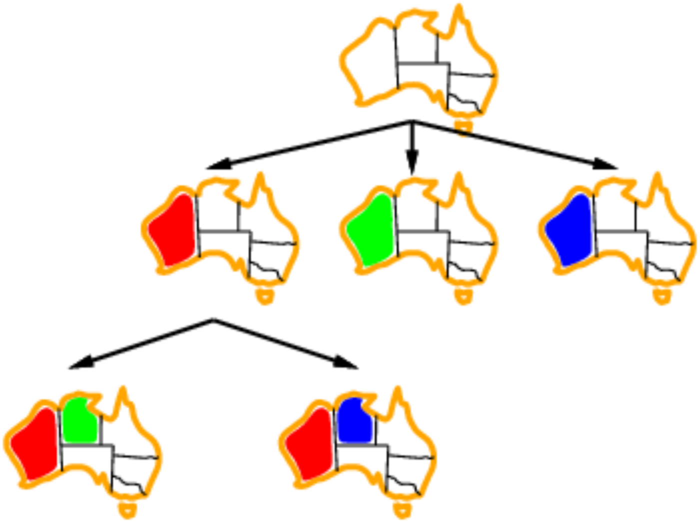


- Solutions: complete & consistent assignments
- Here is one of several solutions
- For generality, constraints can be expressed as relations, e.g., describe WA ≠ NT a
  {(red,green), (red,blue), (green,red), (green,blue), (blue,red),(blue,green)}

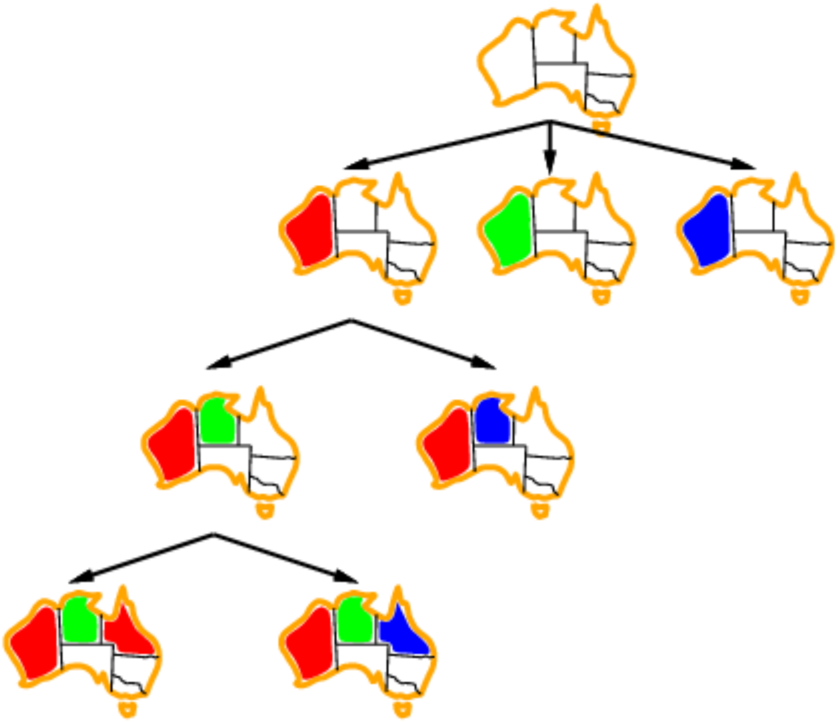
Backtracking example

Backtracking example

Backtracking example



Backtracking example



Basic Backtracking Algorithm

CSP-BACKTRACKING(PartialAssignment a)

- If a is complete then return a
- X ← select an unassigned variable
- D ← select an ordering for the domain of X
- For each value v in D do

  If v is consistent with a then

  - Add (X= v) to a
  - result ← CSP-BACKTRACKING(a)
  - If result ≠ failure then return result
  - Remove (X= v) from a
- Return failure

Start with CSP-BACKTRACKING({})

Note: this is depth first search; can solve n-queens problems for n ~ 25

Problems with backtracking

- Thrashing: keep repeating the same failed variable assignments
- Things that can help avoid this:
  - Consistency checking
  - Intelligent backtracking schemes
- Inefficiency: can explore areas of the search space that aren't likely to succeed
  - Variable ordering can help

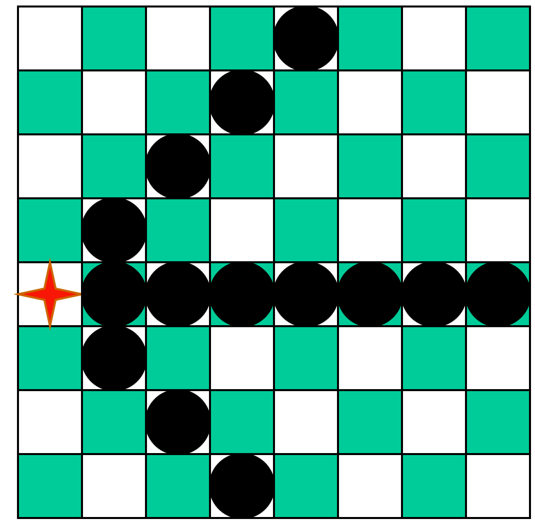
Improving backtracking efficiency

Here are some standard techniques to improve the efficiency of backtracking

- Can we detect inevitable failure early?
- Which variable should be assigned next?
- In what order should its values be tried?

Forward Checking

After variable X is assigned to value v, examine each unassigned variable Y connected to X by a constraint and delete values from Y's domain inconsistent with v

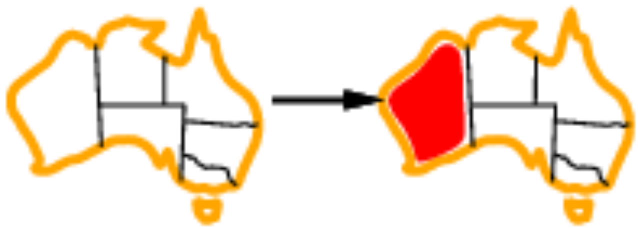


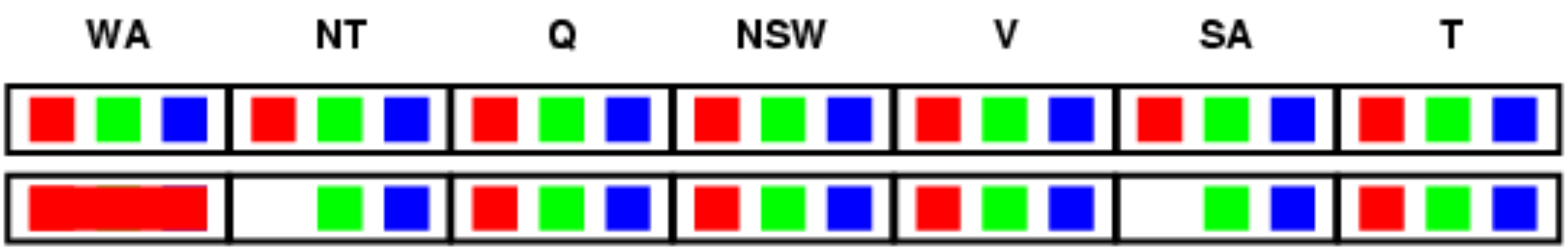
Using forward checking and backward checking roughly doubles the size of N-queens problems that can be practically solved

Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values

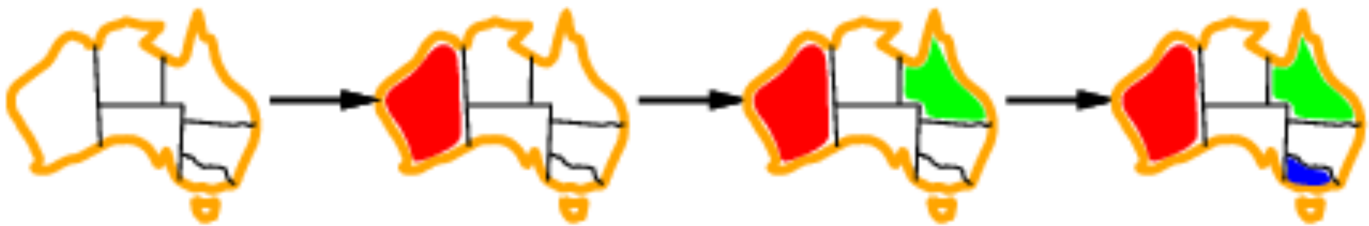
Forward checking





Forward checking

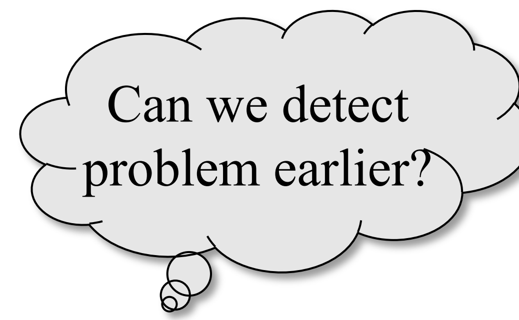
Forward checking



SA (South Australia) domain is empty!

Constraint propagation

- Forward checking propagates info. from assigned to unassigned variables, but doesn't provide early detection for all failures
- NT and SA cannot both be blue!

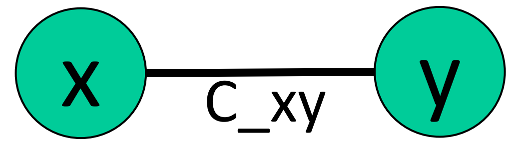


Definition: Arc consistency

- A constraint C_xy is arc consistent w.r.t. x if for each value v of x there is an allowed value of y
- Similarly define C_xy as arc consistent w.r.t. y
- Binary CSP is arc consistent iff every constraint C_xy is arc consistent w.r.t. x as well as y
- When a CSP is not arc consistent, we can make it arc consistent by using the AC3 algorithm
  - Also called "enforcing arc consistency"

Arc Consistency Example 1

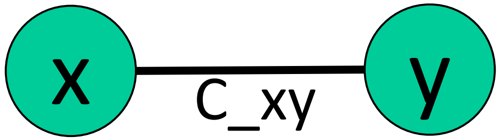
- Domains
  - D_x = {1, 2, 3}
  - D_y = {3, 4, 5, 6}



- Constraint
  - Note: for finite domains, we can represent a constraint as an set of legal value pairs
  - C_xy = {(1,3), (1,5), (3,3), (3,6)}
- C_xy isn’t arc consistent w.r.t. x or y. By enforcing arc consistency, we get reduced domains
  - D'_x = {1, 3}
  - D'_y={3, 5, 6}

Arc Consistency Example 2

- Domains
  - D_x = {1, 2, 3}
  - D_y = {1, 2, 3}



- Constraint
  - C_xy = `lambda v1,v2: v1 < v2`
- C_xy is not arc consistent w.r.t. x or y. By enforcing arc consistency, we get reduced domains:
  - D'_x = {1, 2}
  - D'_y={2, 3}

Aside: Python lambda expressions

Previous slide expressed constraint between two variables as an anonymous Python function taking two arguments

```
lambda v1,v2: v1 < v2
```

```
>>> f = lambda v1,v2: v1 < v2
>>> f
<function <lambda> at 0x10fcf21e0>
>>> f(100,200)
True
>>> f(200,100)
False
```

Python uses lambda after Alonzo Church's lambda calculus from the 1930s

Arc consistency

- Simplest form of propagation makes each arc consistent
- X → Y is consistent iff for every value x_i of X there is some allowed value y_j in Y

Arc consistency

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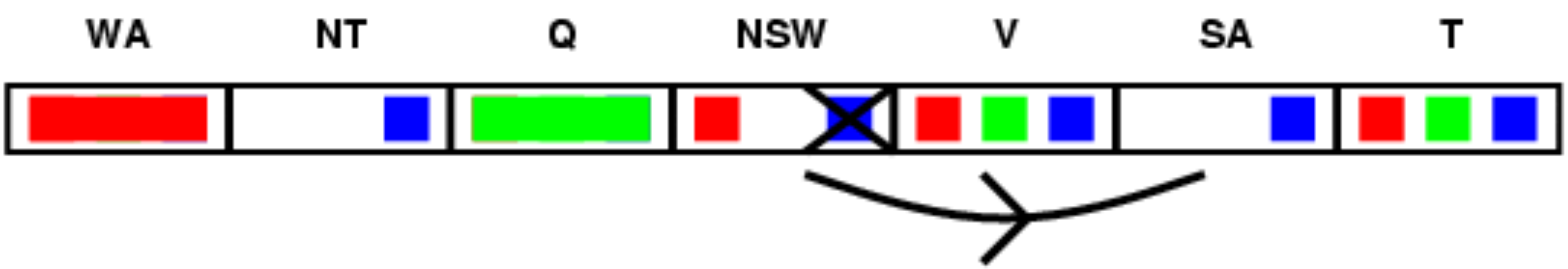
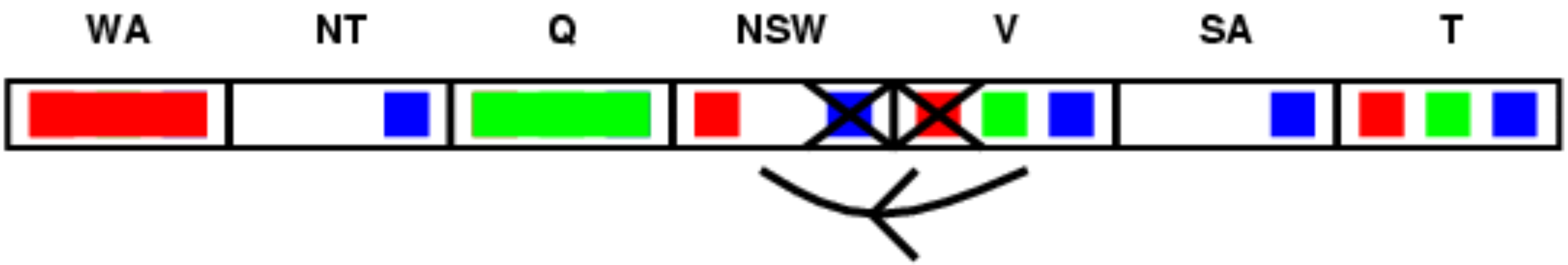


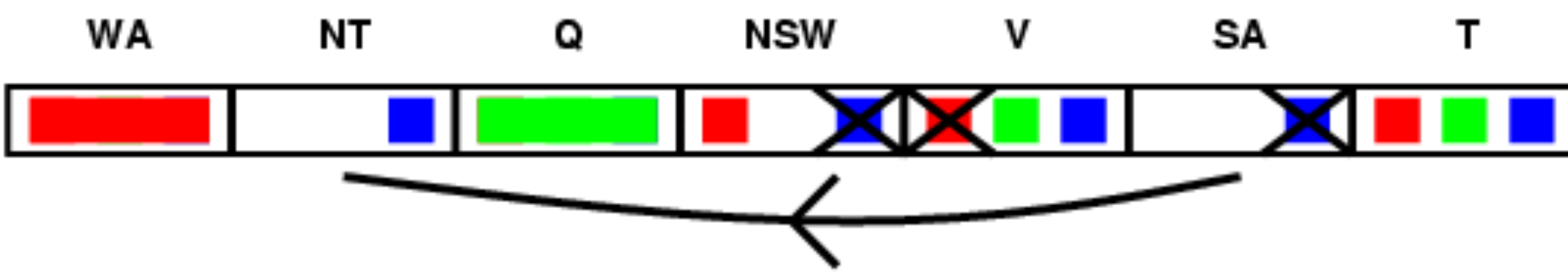
Arc consistency



If X loses a value, neighbors of X need to be rechecked

Arc consistency

- Arc consistency detects failure earlier than simple forward checking
- WA=red and Q=green is quickly recognized as a **deadend**, i.e. an impossible partial instantiation
- The arc consistency algorithm can be run as a preprocessor or after each assignment



General CP for Binary Constraints

Algorithm AC3

contradiction ← false

Q ← stack of all variables

while Q is not empty and not contradiction do

X ← UNSTACK(Q)

For every variable Y adjacent to X do

If REMOVE-ARC-INCONSISTENCIES(X,Y)

If domain(Y) is non-empty then STACK(Y,Q)

else return false

Complexity of AC3

- e = number of constraints (edges)
- d = number of values per variable
- Each variable is inserted in queue up to d times
- REMOVE-ARC-INCONSISTENCY takes $O(d^2)$ time
- CP takes $O(ed^3)$ time

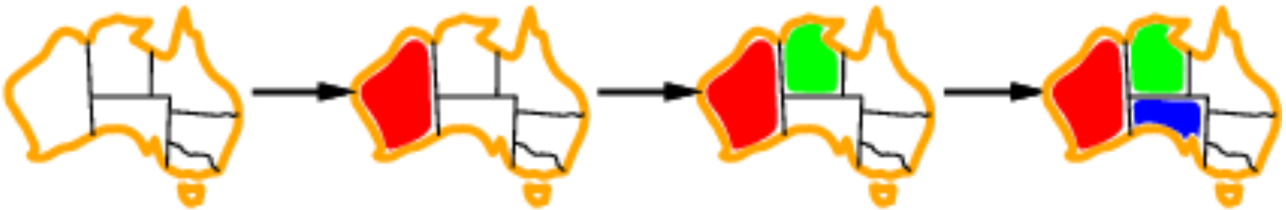
Improving backtracking efficiency

- Some standard techniques to improve the efficiency of backtracking
  - Can we detect inevitable failure early?
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Combining constraint propagation with these heuristics makes 1000-queen puzzles feasible

Most constrained variable

- Most constrained variable:

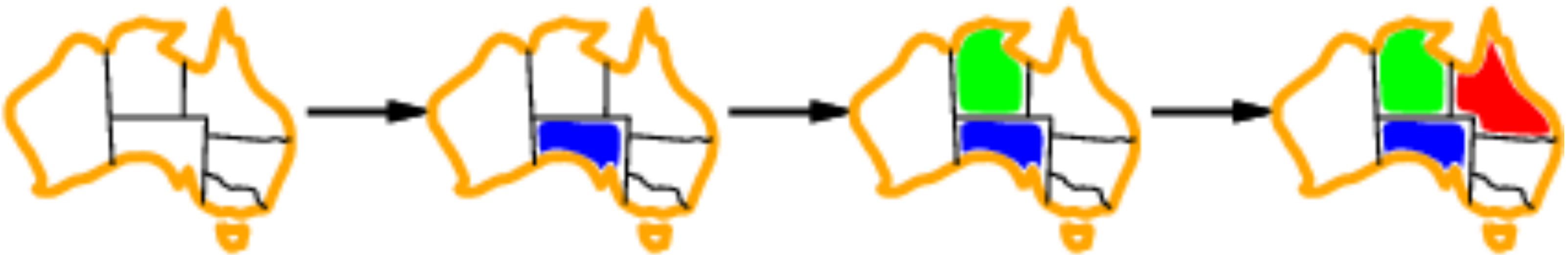
  choose the variable with the fewest legal values



- a.k.a. minimum remaining values (MRV) heuristic
- After assigning value to WA, both NT and SA have only two values in their domains
  – choose one of them rather than Q, NSW, V or T

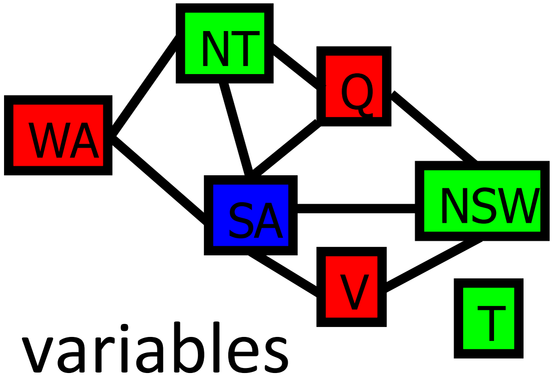
Most constraining variable

- Tie-breaker among most constrained variables
- Choose variable involved in largest # of constraints on remaining variables

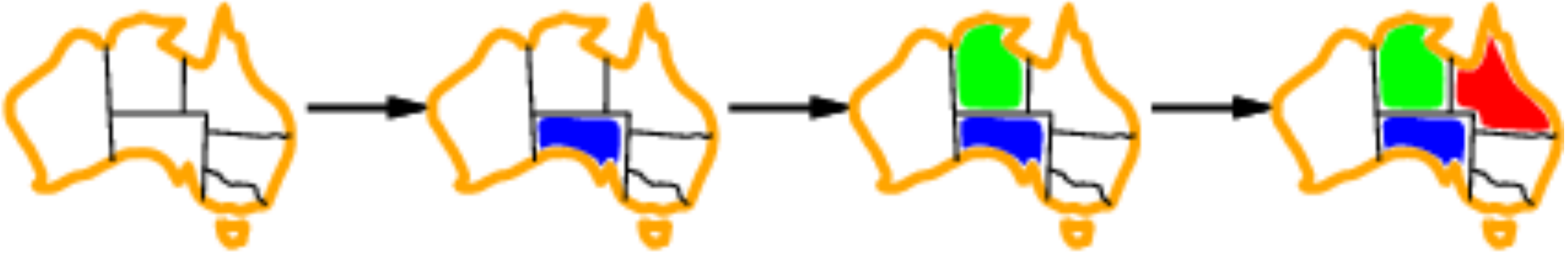


- After assigning SA to be blue, WA, NT, Q, NSW and V all have just two values left.
- WA and V have only one constraint on remaining variables and T none, so choose one of NT, Q & NSW

Most constraining variable



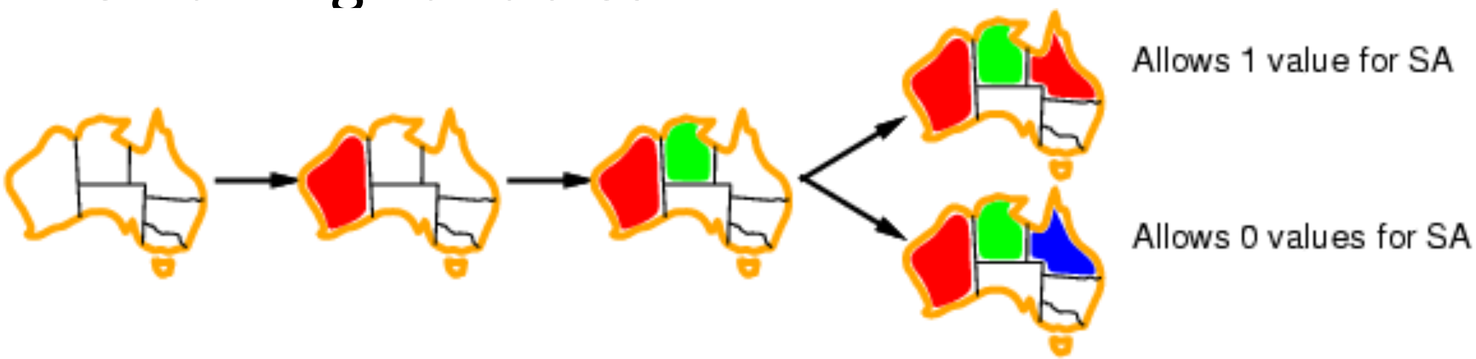
- Tie-breaker among most constrained variables
- Choose variable involved in largest # of constraints on remaining variables



- After assigning SA to be blue, WA, NT, Q, NSW and V all have just two values left.
- WA and V have only one constraint on remaining variables and T none, so choose one of NT, Q & NSW

Least constraining value

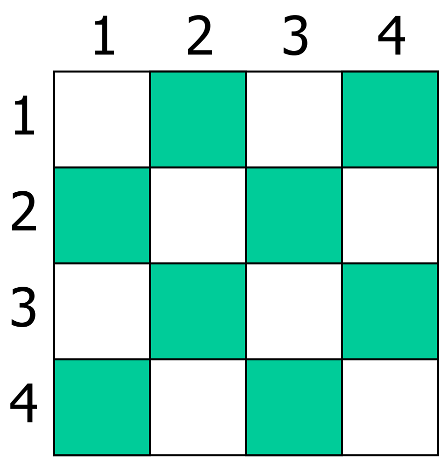
- Given a variable, choose least constraining value:
  - the one that rules out the fewest values in the remaining variables

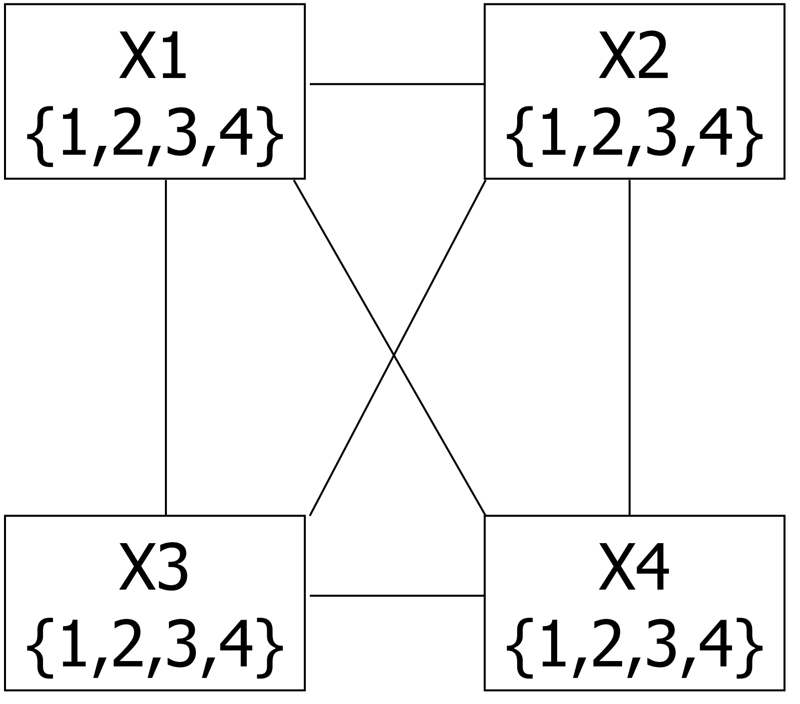


- Combining these heuristics makes 1000 queens feasible
- What's an intuitive explanation for this?

Is AC3 Alone Sufficient?

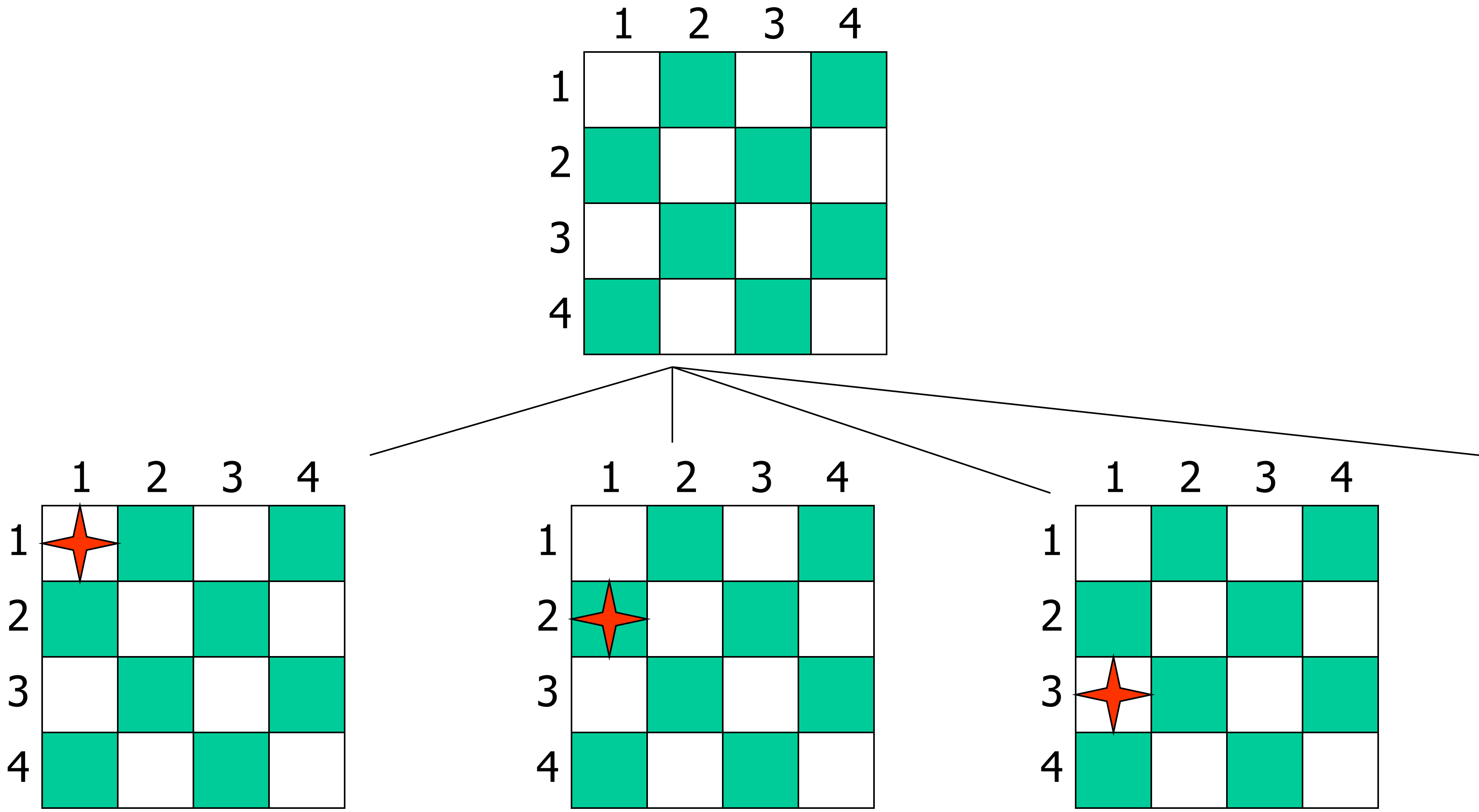
Consider the four queens problem





Solving a CSP still requires search

- Search:
  - can find good solutions, but must examine non-solutions along the way
- Constraint Propagation:
  - can rule out non-solutions, but this is not the same as finding solutions
- Interweave constraint propagation & search:
  - perform constraint propagation at each search step



1 2 3 4
1
2
3
4
1 2 3 4
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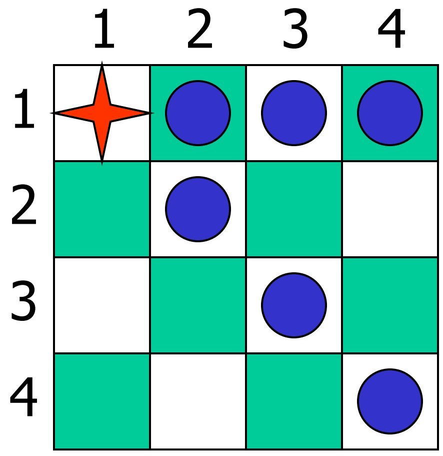
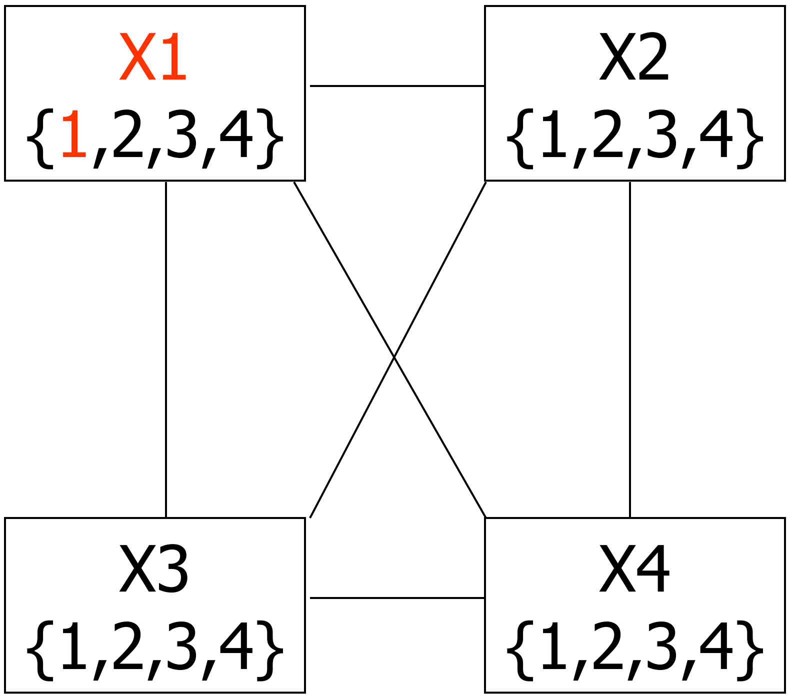
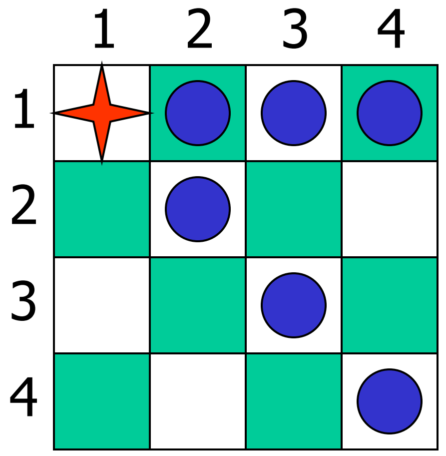
4

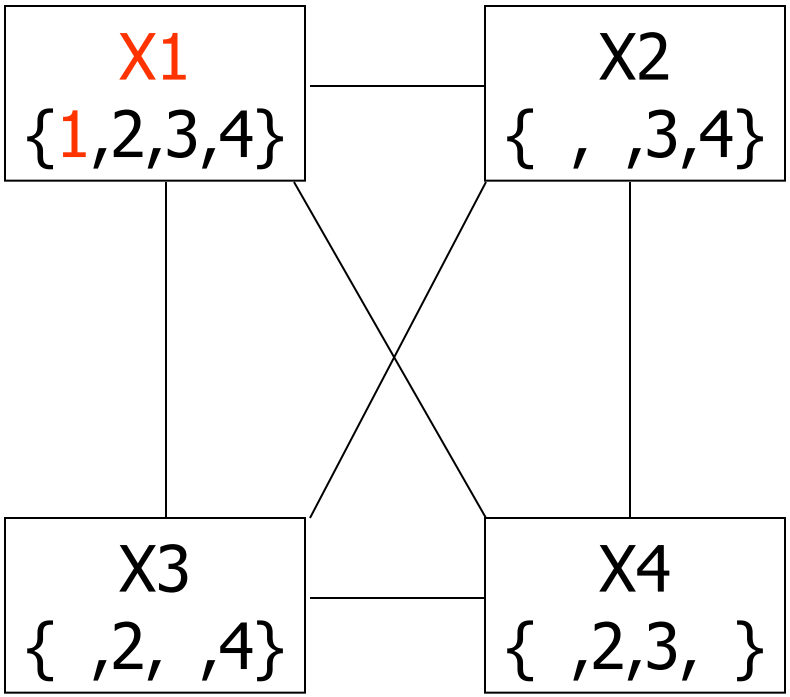
4-Queens Problem



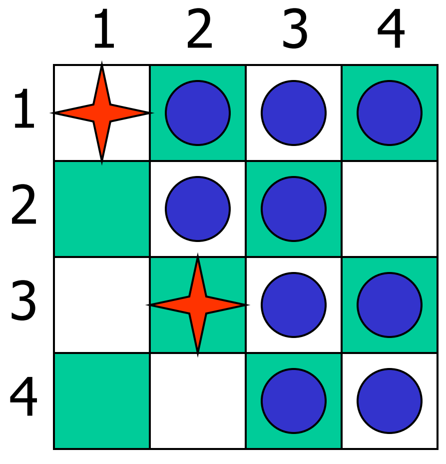


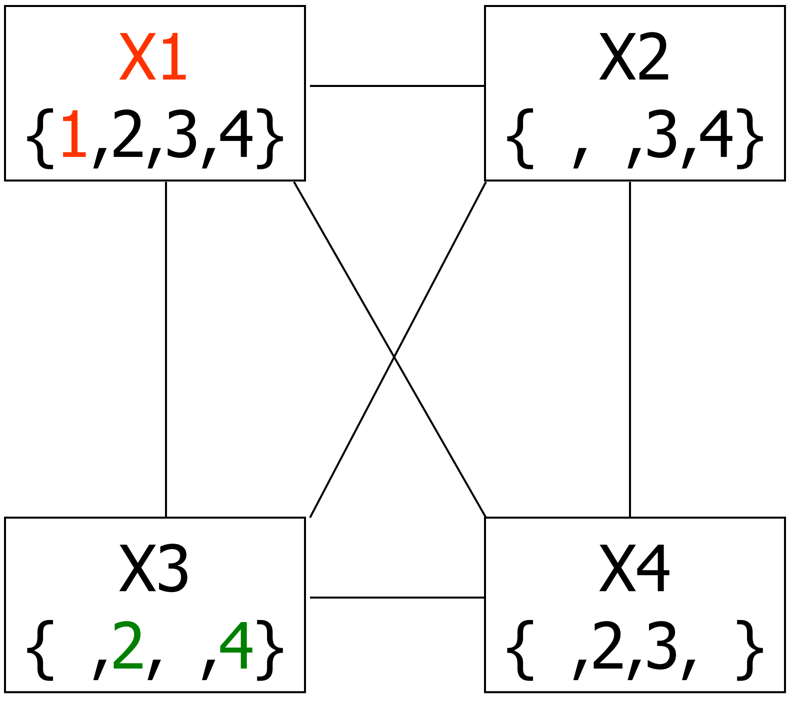
4-Queens Problem





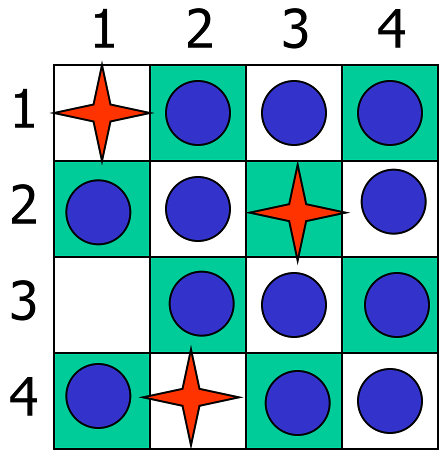
4-Queens Problem

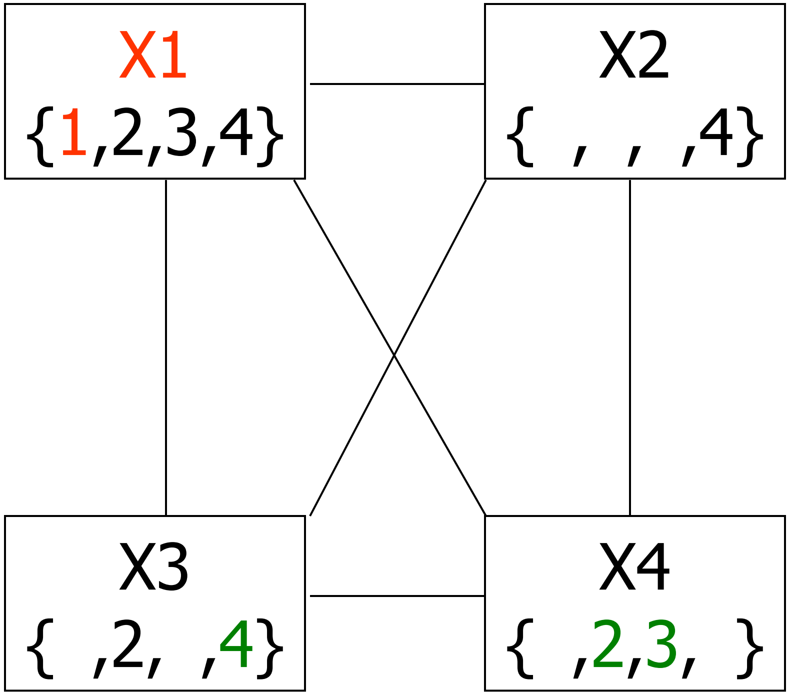




X2=3 eliminates { X3=2, X3=3, X3=4 }
⇒ inconsistent!

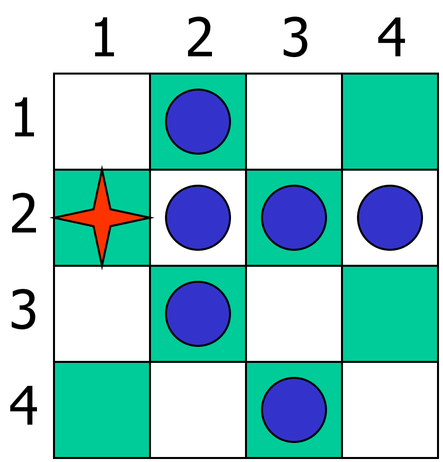
4-Queens Problem

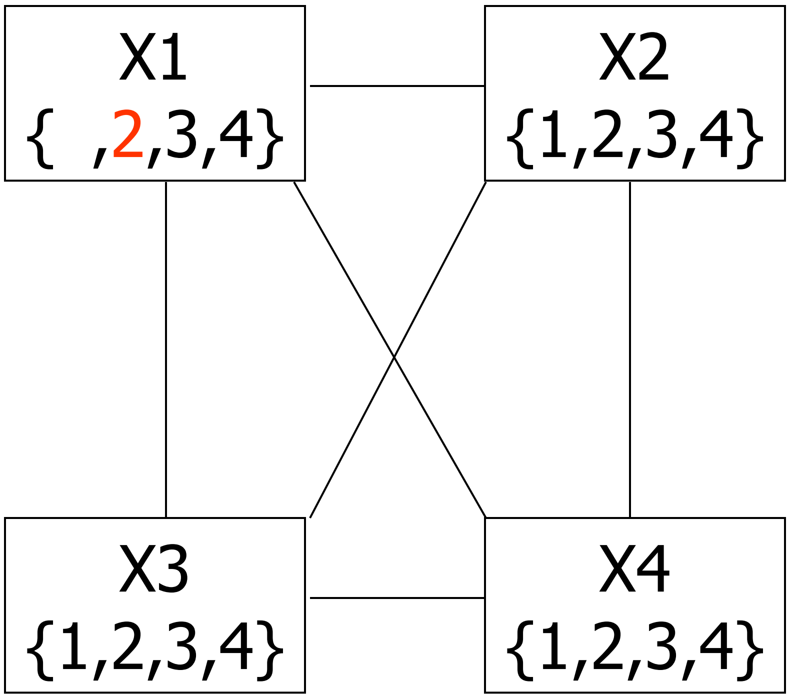




X2=4 ⇒ X3=2, which eliminates { X4=2, X4=3}
⇒ inconsistent!

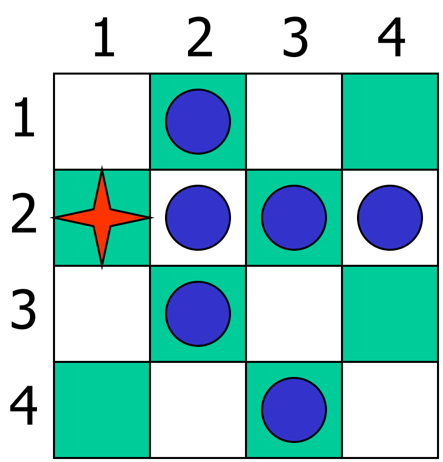
4-Queens Problem

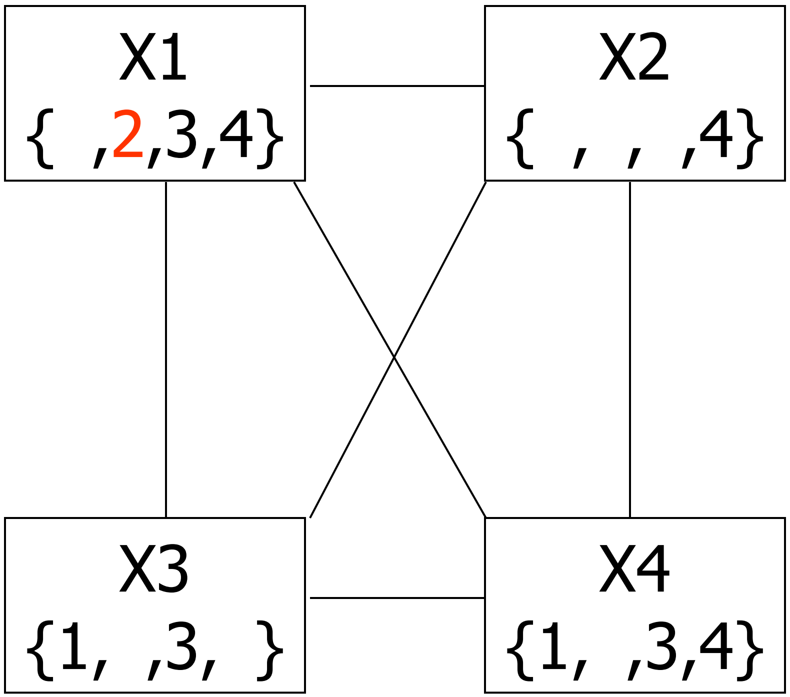




X1 can't be 1, let's try 2

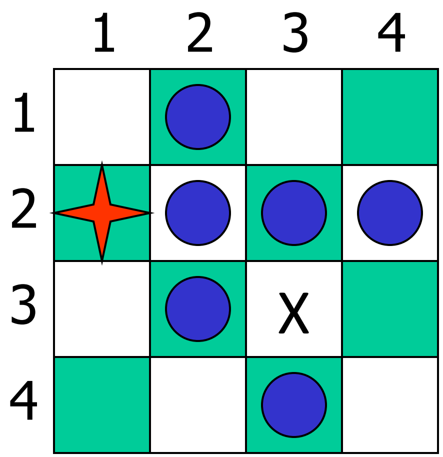
4-Queens Problem

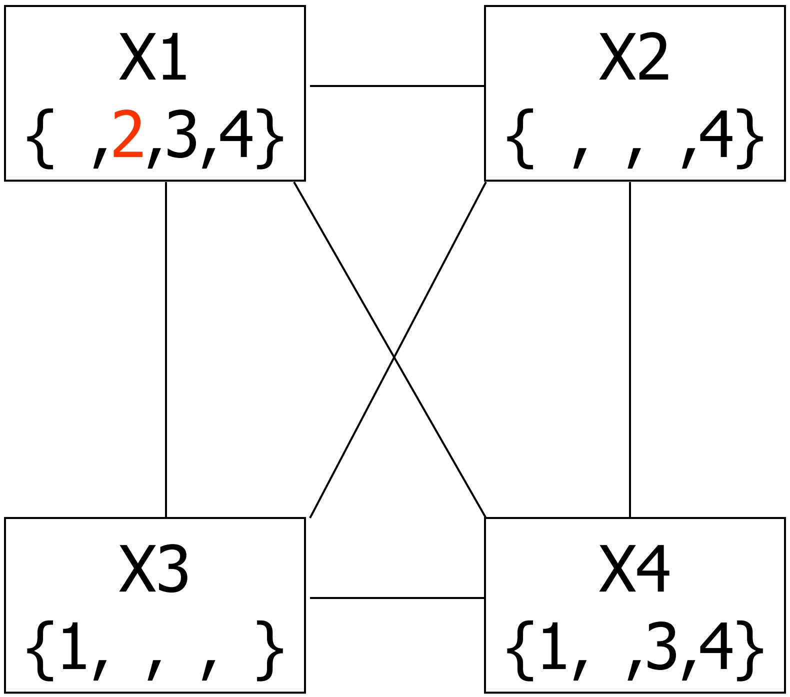




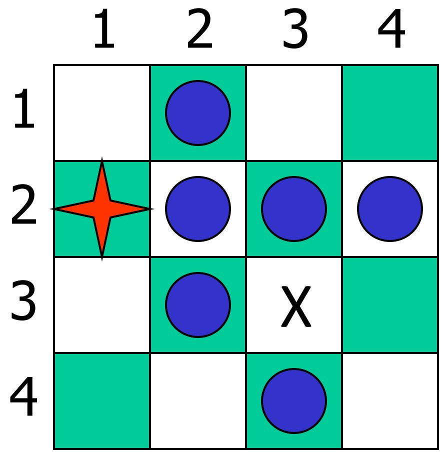
Can we eliminate any other values?

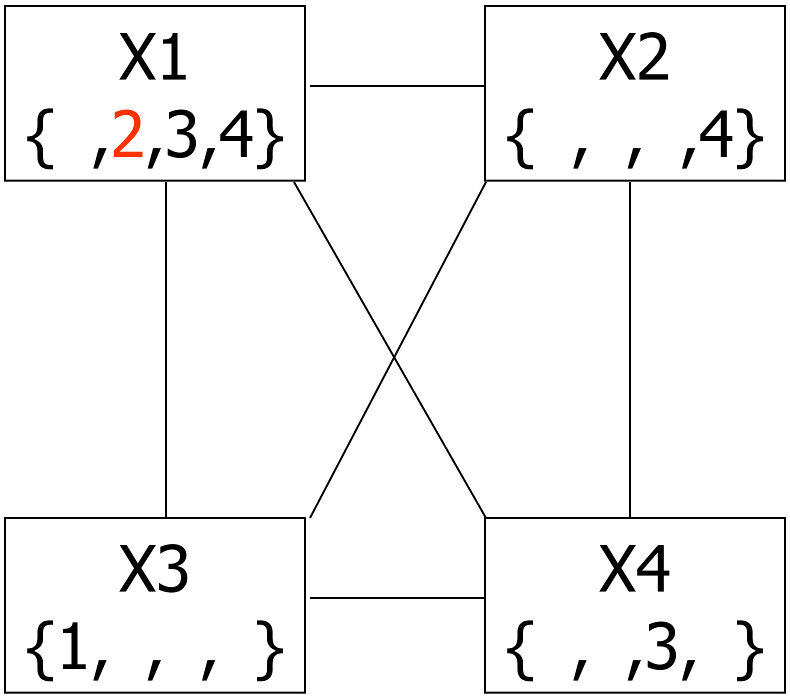
4-Queens Problem





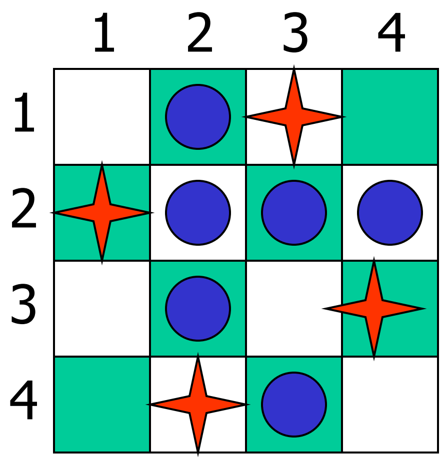
4-Queens Problem

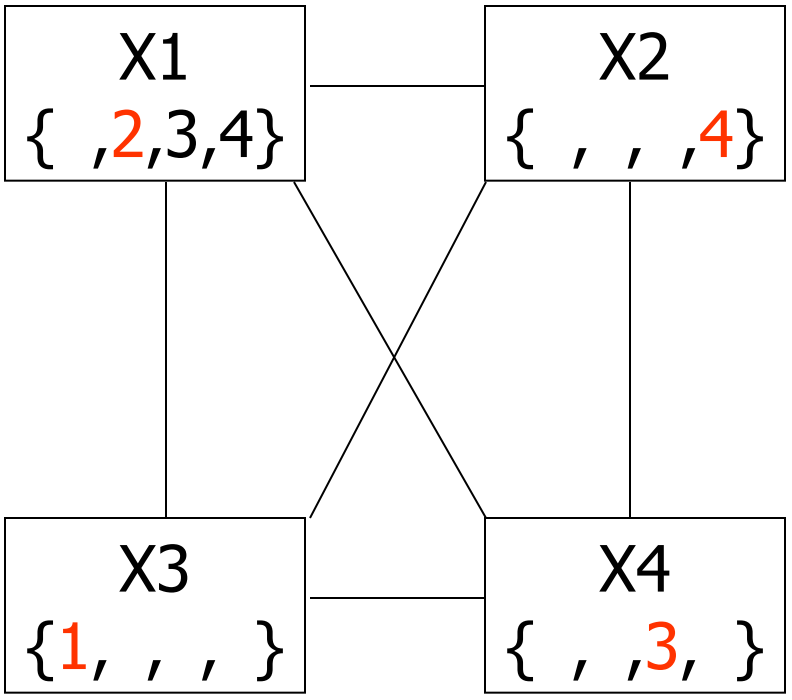




Arc constancy eliminates x3=3 because it's not consistent with X2's remaining values

4-Queens Problem





There is only one solution with X1=2

Sudoku Example

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

initial problem

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

a solution

How can we set this up as a CSP?

Sudoku

- Digit placement puzzle on 9x9 grid with unique answer
- Given an initial partially filled grid, fill remaining squares with a digit between 1 and 9
- Each column, row, and nine 3 × 3 sub-grids must contain all nine digits

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

- Some initial configurations are easy to solve and some very difficult

```python
def sudoku(initValue):
    p = Problem()
    # Define a variable for each cell: 11,12,13...21,22,23...98,99
    for i in range(1, 10) :
        p.addVariables(range(i*10+1, i*10+10), range(1, 10))
    # Each row has different values
    for i in range(1, 10) :
        p.addConstraint(AllDifferentConstraint(), range(i*10+1, i*10+10))
    # Each column has different values
    for i in range(1, 10) :
        p.addConstraint(AllDifferentConstraint(), range(10+i, 100+i, 10))
    # Each 3x3 box has different values
    p.addConstraint(AllDifferentConstraint(), [11,12,13,21,22,23,31,32,33])
    p.addConstraint(AllDifferentConstraint(), [41,42,43,51,52,53,61,62,63])
    p.addConstraint(AllDifferentConstraint(), [71,72,73,81,82,83,91,92,93])

    p.addConstraint(AllDifferentConstraint(), [14,15,16,24,25,26,34,35,36])
    p.addConstraint(AllDifferentConstraint(), [44,45,46,54,55,56,64,65,66])
    p.addConstraint(AllDifferentConstraint(), [74,75,76,84,85,86,94,95,96])

    p.addConstraint(AllDifferentConstraint(), [17,18,19,27,28,29,37,38,39])
    p.addConstraint(AllDifferentConstraint(), [47,48,49,57,58,59,67,68,69])
    p.addConstraint(AllDifferentConstraint(), [77,78,79,87,88,89,97,98,99])

    # add unary constraints for cells with initial non-zero values
    for i in range(1, 10) :
        for j in range(1, 10):
            value = initValue[i-1][j-1]
            if value:
                p.addConstraint(lambda var, val=value: var == val, (i*10+j,))
    return p.getSolution()
```

```python
# Sample problems
easy = [
  [0,9,0,7,0,0,8,6,0],
  [0,3,1,0,0,5,0,2,0],
  [8,0,6,0,0,0,0,0,0],
  [0,0,7,0,5,0,0,0,6],
  [0,0,0,3,0,7,0,0,0],
  [5,0,0,0,1,0,7,0,0],
  [0,0,0,0,0,0,1,0,9],
  [0,2,0,6,0,0,0,5,0],
  [0,5,4,0,0,8,0,7,0]]

hard = [
  [0,0,3,0,0,0,4,0,0],
  [0,0,0,0,7,0,0,0,0],
  [5,0,0,4,0,6,0,0,2],
  [0,0,4,0,0,0,8,0,0],
  [0,9,0,0,3,0,0,2,0],
  [0,0,7,0,0,0,5,0,0],
  [6,0,0,5,0,2,0,0,1],
  [0,0,0,0,9,0,0,0,0],
  [0,0,9,0,0,0,3,0,0]]

very_hard = [
  [0,0,0,0,0,0,0,0,0],
  [0,0,9,0,6,0,3,0,0],
  [0,7,0,3,0,4,0,9,0],
  [0,0,7,2,0,8,6,0,0],
  [0,4,0,0,0,0,0,7,0],
  [0,0,2,1,0,6,5,0,0],
  [0,1,0,9,0,5,0,4,0],
  [0,0,8,0,2,0,7,0,0],
  [0,0,0,0,0,0,0,0,0]]
```

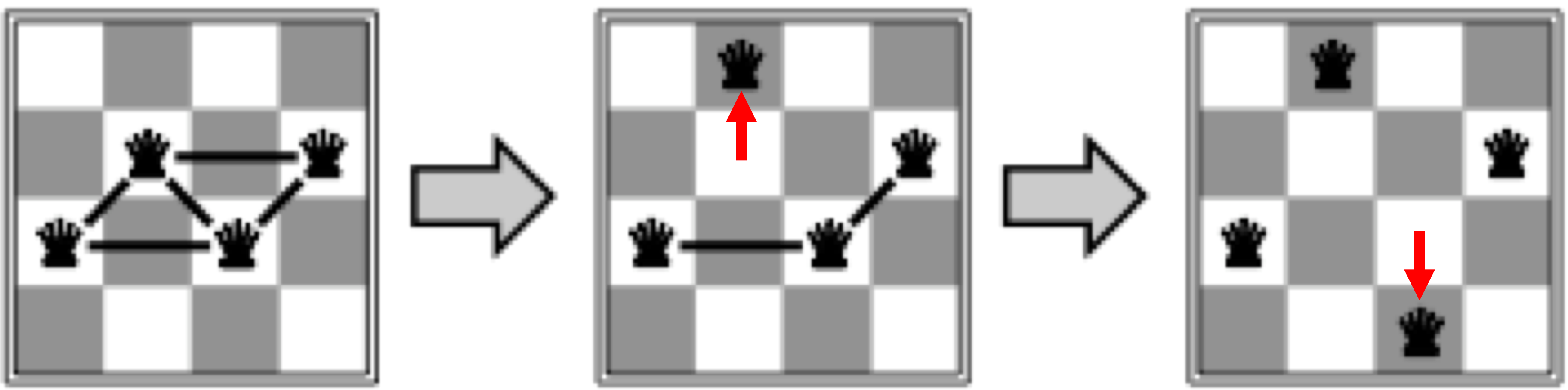
Local search for constraint problems

- Remember local search?
- There's a version of local search for CSP problems
- Basic idea:
  - generate a random "solution"
  - Use metric of "number of conflicts"
  - Modifying solution by reassigning one variable at a time to decrease metric until solution found or no modification improves it
- Has all features and problems of local search

  like....?

Min Conflict Example

- **States:** 4 Queens, 1 per column
- **Operators:** Move a queen in its column
- **Goal test:** No attacks
- **Evaluation metric:** Total number of attacks



How many conflicts does each state have?

Basic Local Search Algorithm

Assign one domain value d_i to each variable v_i

while no solution & not stuck & not timed out:

    bestCost $\leftarrow \infty$;   bestList $\leftarrow$ [ ];

    for each variable v_i | Cost(Value(v_i)) > 0

        for each domain value d_i of v_i

            if Cost(d_i) < bestCost

                bestCost $\leftarrow$ Cost(d_i); bestList $\leftarrow$ [d_i];

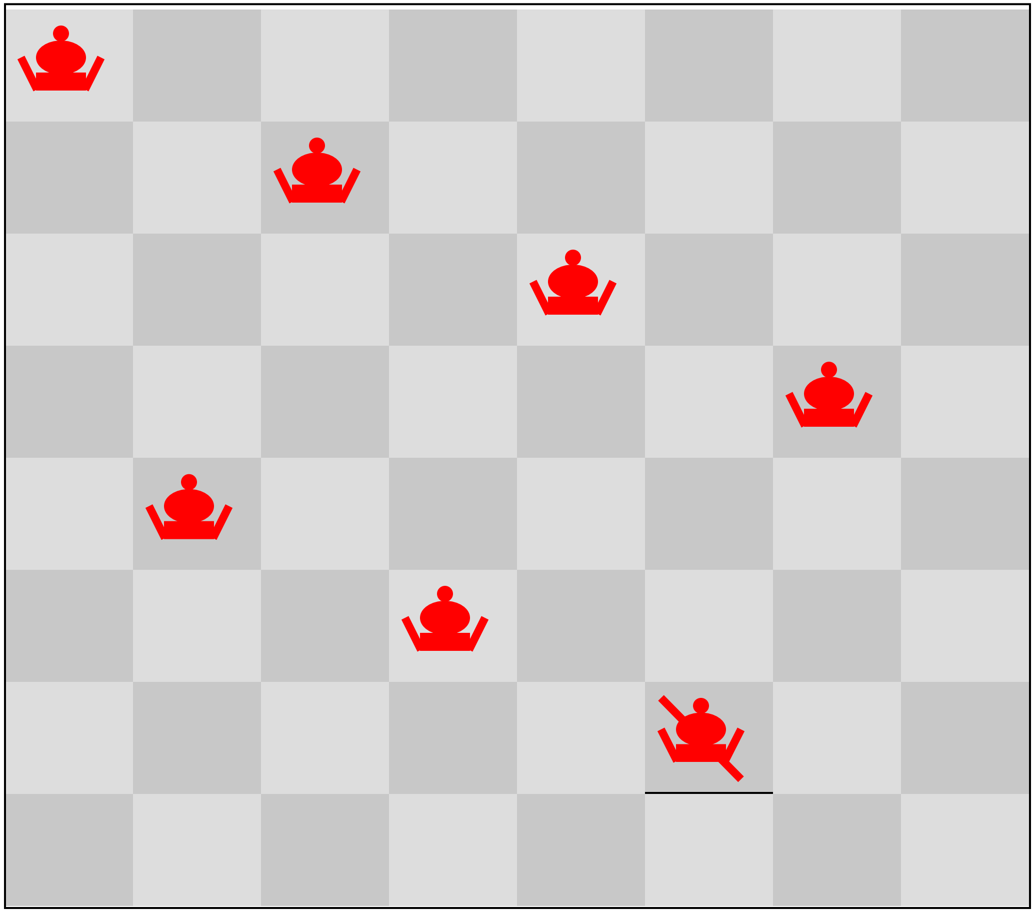
            else if Cost(d_i) = bestCost

                bestList $\leftarrow$ bestList $\cup$ d_i

    Take a randomly selected move from bestList

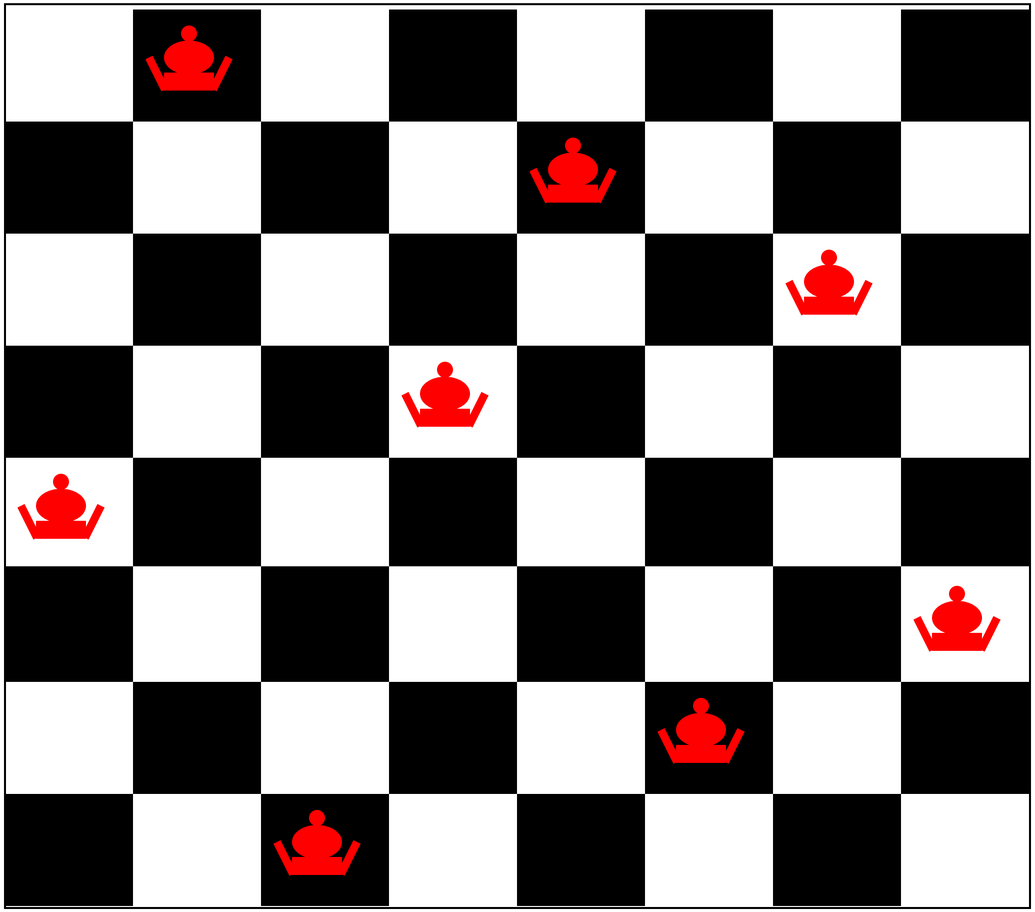
Eight Queens using Backtracking

Undo move
for Queen 7
and so on...

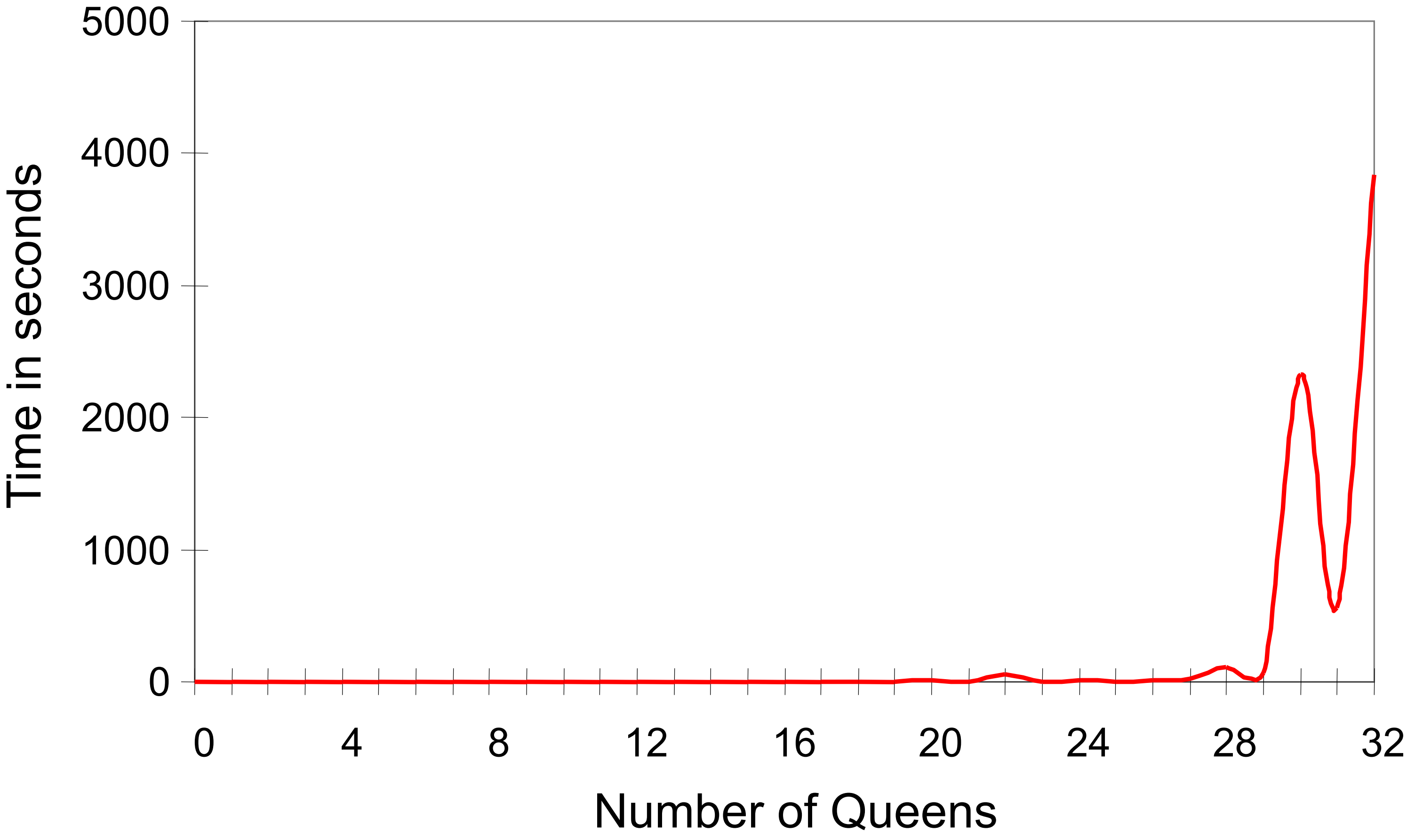


Eight Queens using Local Search

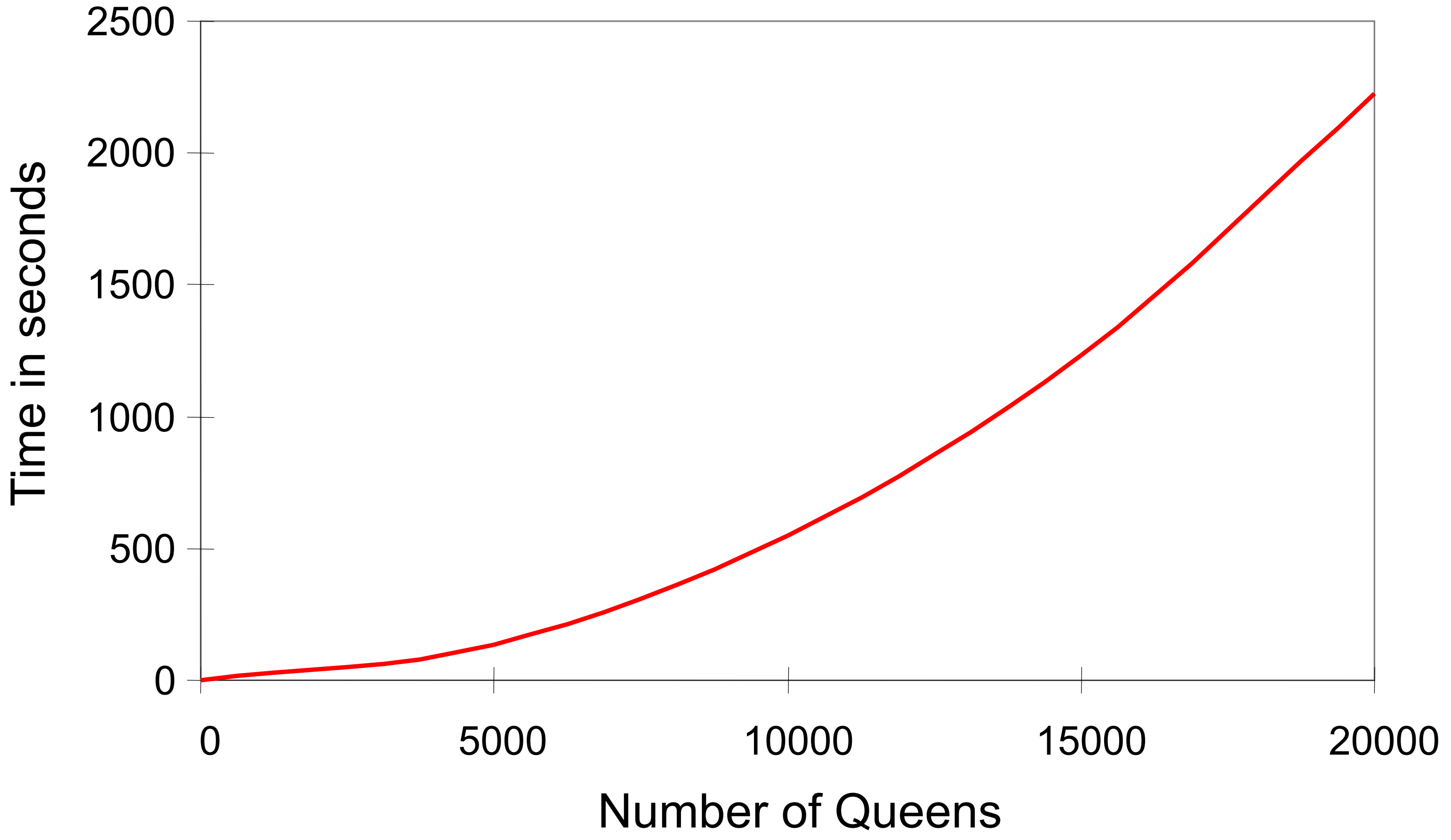
Answer Found



Backtracking Performance



Local Search Performance

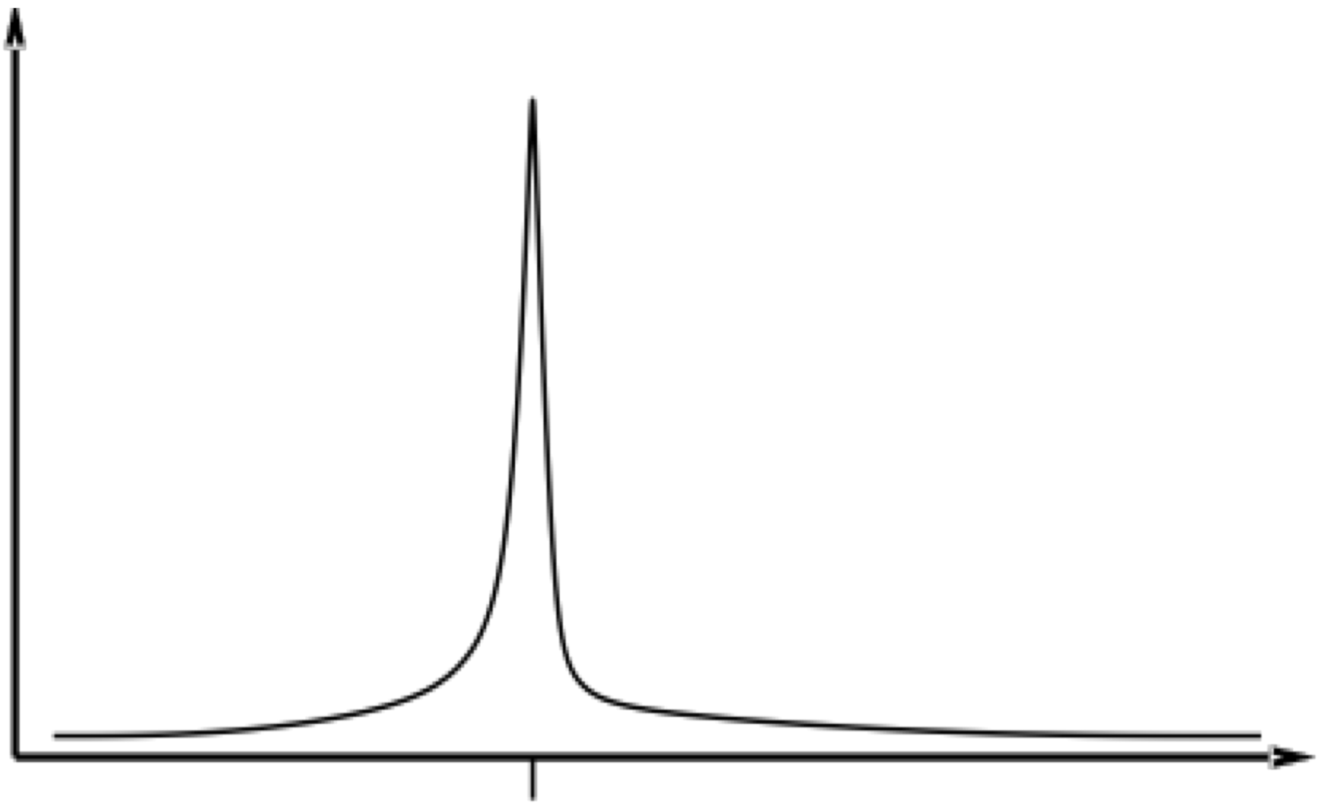


Min Conflict Performance

- Performance depends on quality and informativeness of initial assignment; inversely related to distance to solution
- Min Conflict often has astounding performance
- Can solve arbitrary size (i.e., millions) N-Queens problems in constant time
- Appears to hold for arbitrary CSPs with the caveat…

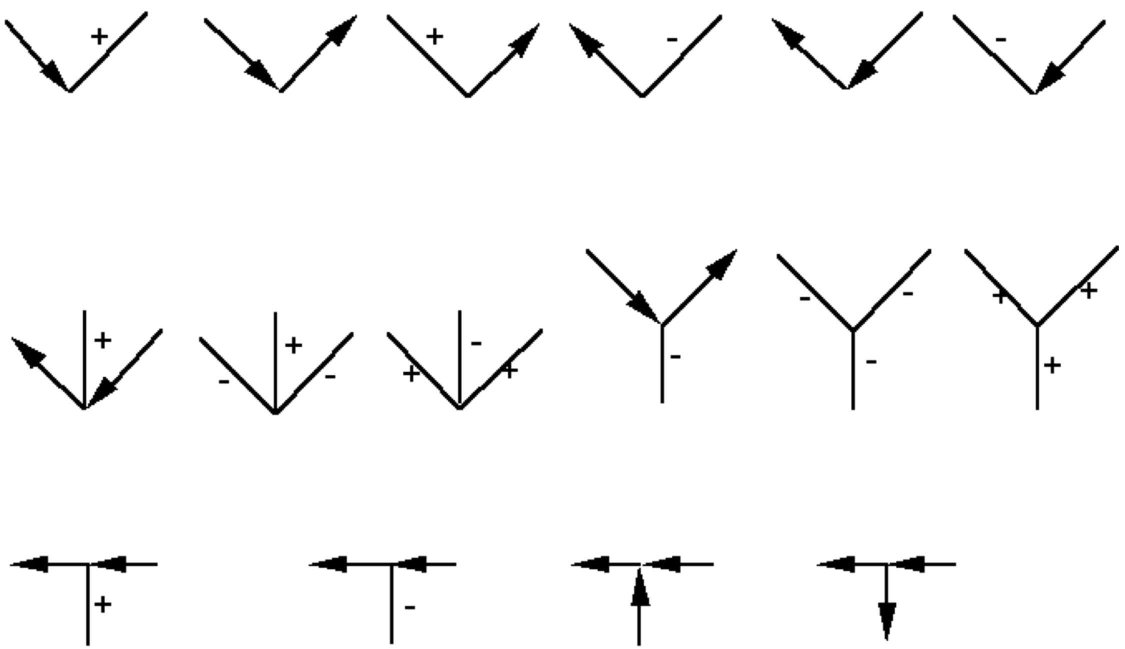
Min Conflict Performance

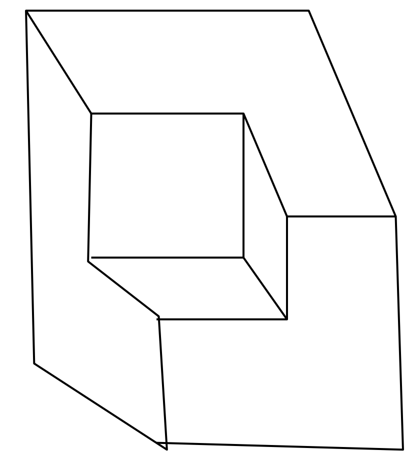
Except in a certain critical range of the ratio constraints to variables.



Famous example: labeling line drawings

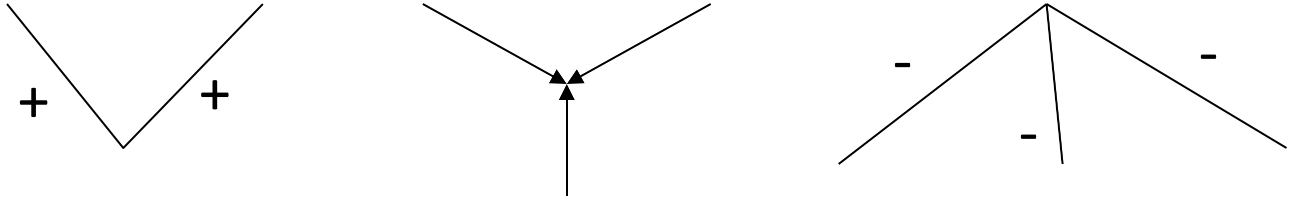
- Waltz labeling algorithm, earliest AI CSP application (1972)
  - Convex interior lines labeled as +
  - Concave interior lines labeled as –
  - Boundary lines labeled as → with background to left
- 208 labeling possible labelings, but only 18 are legal





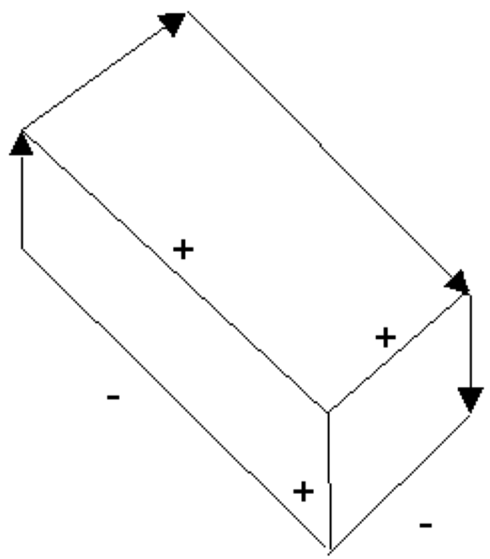
Labeling line drawings II

Here are some illegal labelings

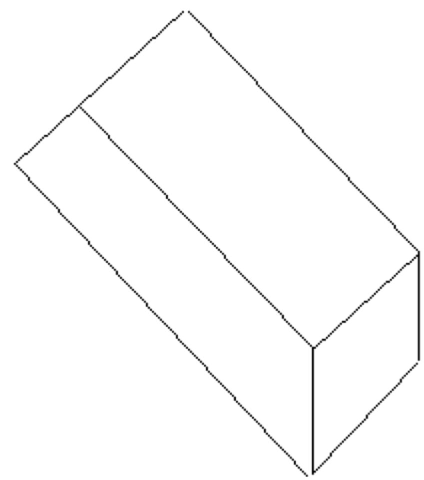


Labeling line drawings

Waltz labeling algorithm: propagate constraints repeatedly until a solution is found

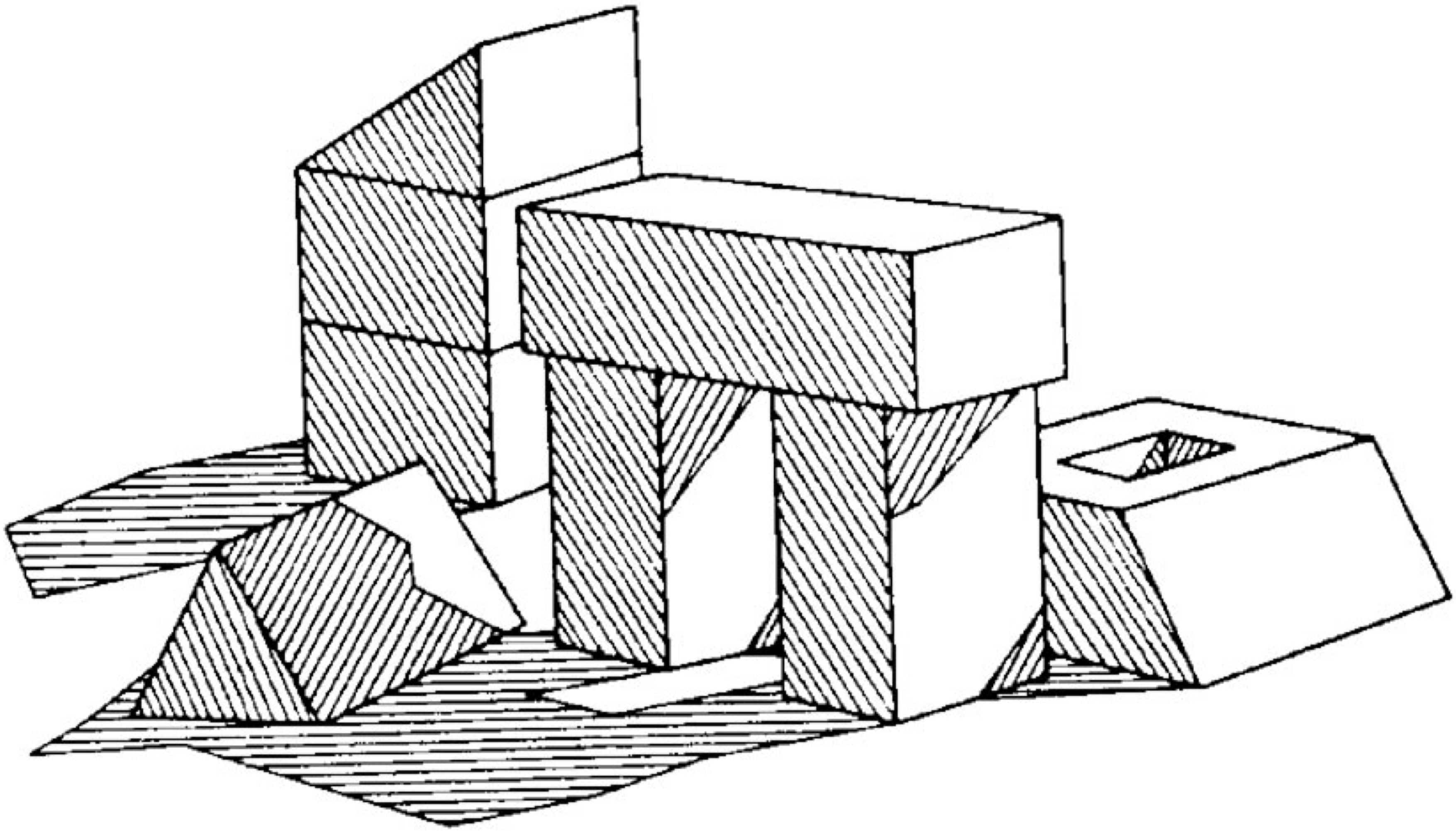


solution for one labeling problem



labeling problem with no solution

Shadows add complexity



CSP was able to label scenes where some of the lines were caused by shadows

Challenges for constraint reasoning

- What if not all constraints can be satisfied?
  - Hard vs. soft constraints vs. preferences
  - Degree of constraint satisfaction
  - Cost of violating constraints
- What if constraints are of different forms?
  - Symbolic constraints
  - Logical constraints
  - Numerical constraints [constraint solving]
  - Temporal constraints
  - Mixed constraints

Challenges for constraint reasoning

- What if constraints are represented intentionally?
  - Cost of evaluating constraints (time, memory, resources)
- What if constraints, variables, and/or values change over time?
  - Dynamic constraint networks
  - Temporal constraint networks
  - Constraint repair
- What if multiple agents or systems are involved in constraint satisfaction?
  - Distributed CSPs
  - Localization techniques