1. (10 points) Consider the rod-cutting problem for a rod of length six with the profit for each length of rod given in Table 1.

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_i)</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>12</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 1: Profit \(p_i\) for a rod of length \(i\)

(a) Complete the following table of values for \(r[i]\) and \(s[i]\). Show all work.

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r_i)</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>14</td>
<td>17</td>
</tr>
<tr>
<td>(s_i)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Note: For \(s_4\), \(s_5\), and \(s_6\), the values 4, 4, and 4 are also acceptable.

To compute \(r_4\):

\[
r_4 = \max(p_1 + r_3, p_2 + r_2, p_3 + r_1, p_4 + r_0) = \max(10, 10, 10, 12) = 12
\]

which occurs with an initial cut of four. To compute \(r_5\):

\[
r_5 = \max(p_1 + r_4, p_2 + r_3, p_3 + r_2, p_4 + r_1, p_5 + r_0) = \max(14, 13, 13, 13, 12) = 14
\]

which occurs with an initial cut of one. The computation of \(r_6\) is similar.

(b) Use the \(s\) table to determine the optimal cuts for a rod of length six. Justify your answer.

The value of \(s_6\) is two, so we should make an initial cut of length two, leaving a rod of length four. \(s_4\) is four, so we leave this piece uncut. Thus the optimal cuts for a rod of length six are (2, 4).

(continued on other side)
2. (10 points) I own and operate a small delivery truck; every morning I go to a warehouse, load the truck with items awaiting delivery, and deliver them. My truck can carry $W$ pounds of goods. When I arrive at the warehouse, there are $n$ items $x_1, x_2, \ldots, x_n$ waiting to be delivered, each with weight $w_i$ and value $v_i$, $i = 1, 2, \ldots, n$. There are always more items waiting than I can carry in my truck. I want to maximize the total value of my load.

Does this problem have *optimal substructure*? Explain why or why not. If “yes,” then outline a proof.

Yes, the problem does have optimal substructure. Consider a most valuable load that weighs at most $W$ pounds. If we remove item $j$ from the load, the remaining load must be the most valuable load weighing at most $W - w_j$ made from the $n - 1$ items excluding $j$.

Suppose this were not the case. Then there would be a load of weight at most $W - w_j$ that is more valuable. Adding item $j$ to this load yields a solution to the original problem with greater total value, contradicting the supposition that it was optimal.