CMSC 341

Graphs

Basic Graph Definitions

- A ***graph*** G = (V,E) consists of a finite set of ***vertices***, V, and a finite set of ***edges***, E.
- Each edge is a pair (v,w) where v, w $\in$ V.
  - V and E are sets, so each vertex v $\in$ V is unique, and each edge e $\in$ E is unique.
  - Edges are sometimes called ***arcs*** or ***lines***.
  - Vertices are sometimes called ***nodes*** or ***points***.

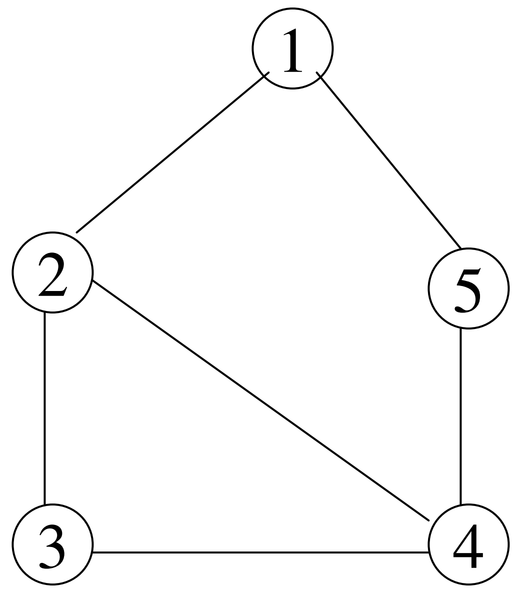
Graph Applications

- Graphs can be used to model a wide range of applications including
- Intersections and streets within a city
- Roads/trains/airline routes connecting cities/countries
- Computer networks
- Electronic circuits

Basic Graph Definitions (2)

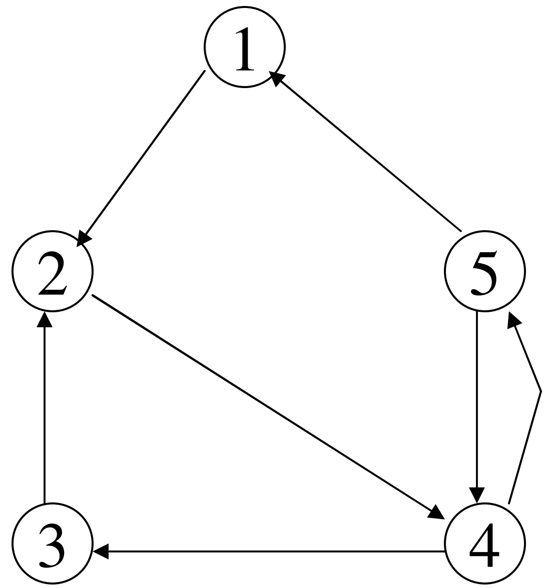
- A ***directed graph*** is a graph in which the edges are ordered pairs.
  That is, $(u,v) \neq (v,u)$, $u, v \in V$.
  Directed graphs are sometimes called ***digraphs***.
- An ***undirected graph*** is a graph in which the edges are unordered pairs.
  That is, $(u,v) = (v,u)$.
- A ***sparse graph*** is one with "few" edges.
  That is $|E| = O(|V|)$
- A ***dense graph*** is one with "many" edges.
  That is $|E| = O(|V|^2)$

Undirected Graph



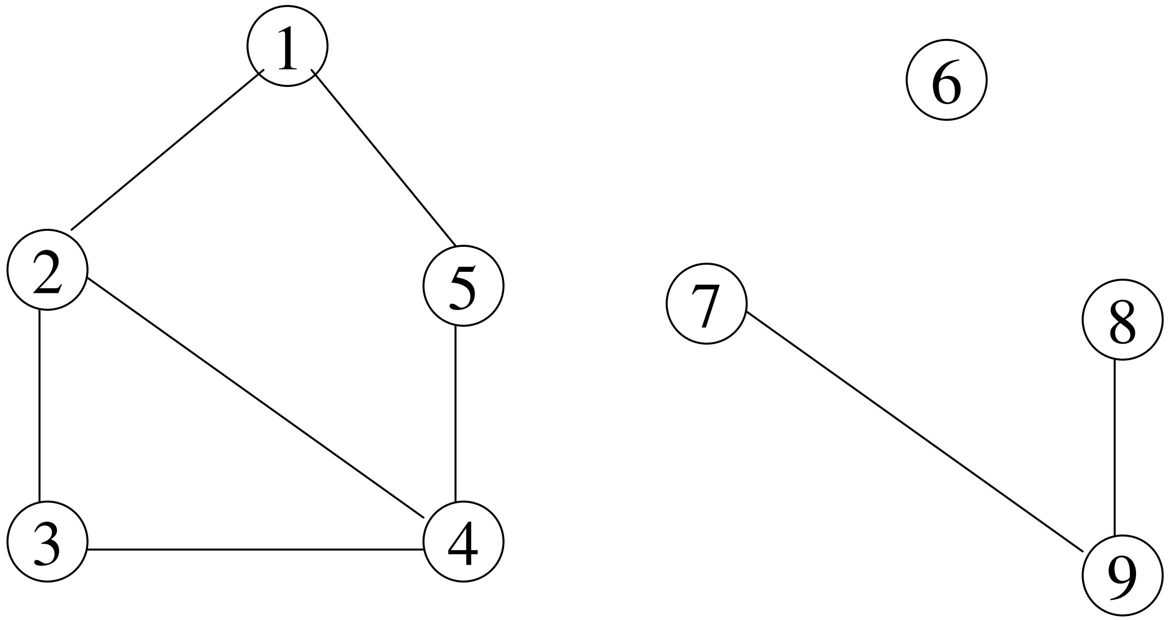
- All edges are two-way. Edges are unordered pairs.
- V = { 1, 2 ,3, 4, 5}
- E = { (1,2), (2, 3), (3, 4), (2, 4), (4, 5), (5, 1) }

Directed Graph



- All edges are “one-way” as indicated by the arrows. Edges are ordered pairs.
- V = { 1, 2, 3, 4, 5}
- E = { (1, 2), (2, 4), (3, 2), (4, 3), (4, 5), (5, 4), (5, 1) }

A Single Graph with Multiple Components

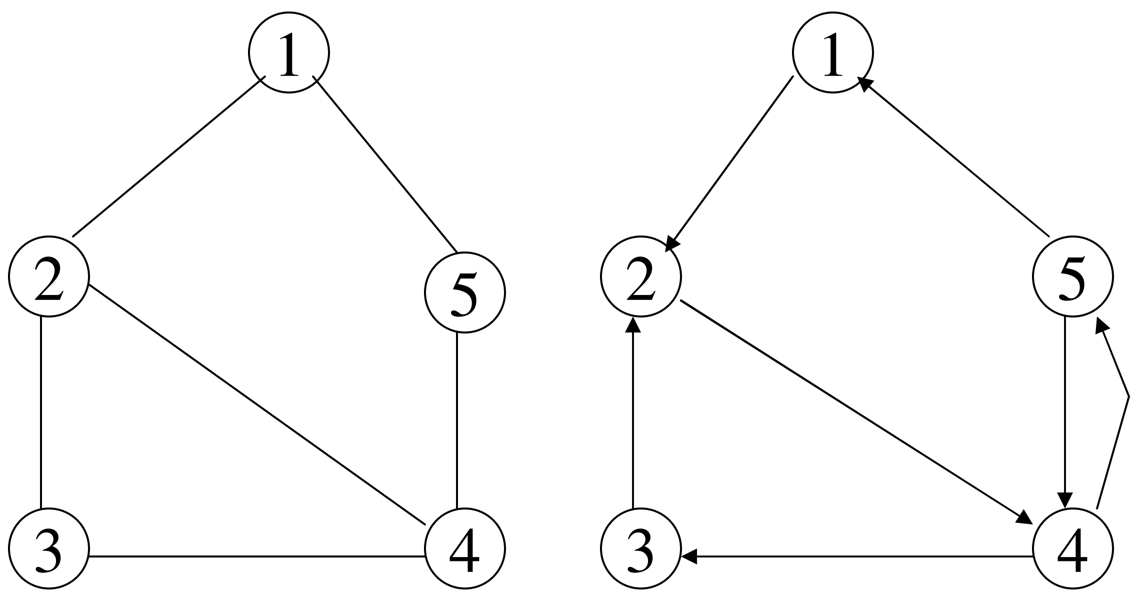


Basic Graph Definitions (3)

- Vertex w is ***adjacent to*** vertex v if and only if (v, w) $\in$ E.
- For undirected graphs, with edge (v, w), and hence also (w, v), w is adjacent to v and v is adjacent to w.
- An edge may also have:
  - ***weight*** or ***cost*** -- an associated value
  - ***label*** -- a unique name
- The ***degree*** of a vertex, v, is the number of vertices adjacent to v. Degree is also called *valence*.

Basic Graph Definitions (4)

- For directed graphs vertex w is ***adjacent to*** vertex v if and only if (v, w) $\in$ E.
- ***Indegree*** of a vertex w is the number of edges (v,w).
- ***OutDegree*** of a vertex w is the number of edges(w,v).

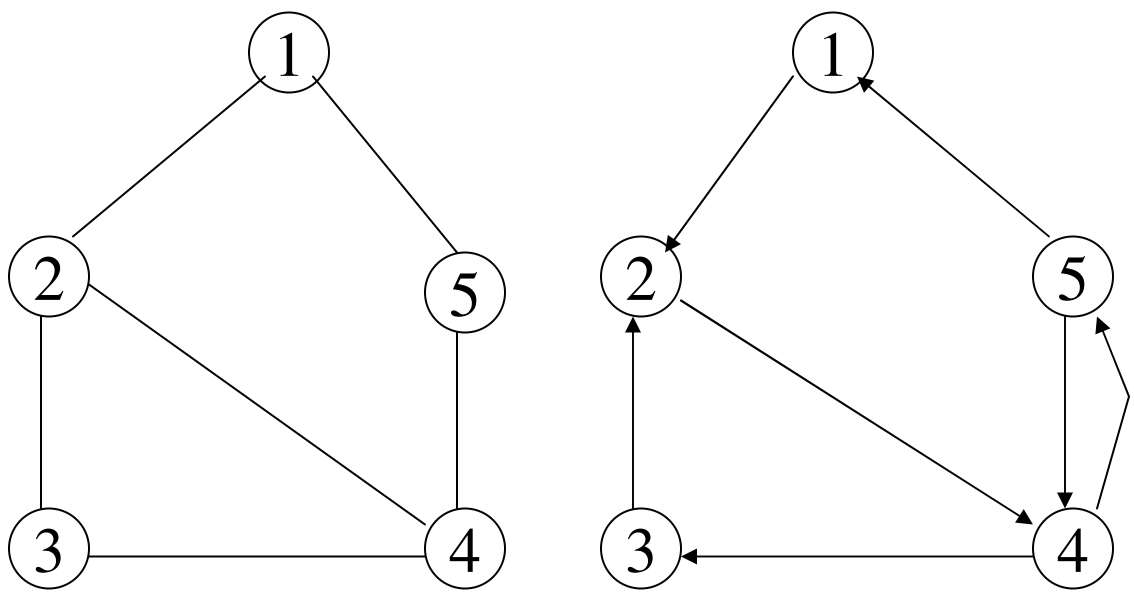


Paths in Graphs

- A ***path*** in a graph is a sequence of vertices $w_1, w_2, w_3, \ldots, w_n$ such that $(w_i, w_{i+1}) \in E$ for $1 \leq i < n$.
- The ***length*** of a path in a graph is the number of edges on the path. The length of the path from a vertex to itself is 0.
- A ***simple path*** is a path such that all vertices are distinct, except that the first and last may be the same.
- A ***cycle*** in a graph is a path $w_1, w_2, w_3, \ldots, w_n$, $w \in V$ such that:
  - there are at least two vertices on the path
  - $w_1 = w_n$ (the path starts and ends on the same vertex)
  - if any part of the path contains the subpath w_i, w_j, w_i, then each of the edges in the subpath is distinct (i. e., no backtracking along the same edge)
- A ***simple cycle*** is one in which the path is simple.
- A directed graph with no cycles is called a ***directed acyclic graph***, often abbreviated as DAG

Paths in Graphs (2)

- How many simple paths from 1 to 4 and what are their lengths?



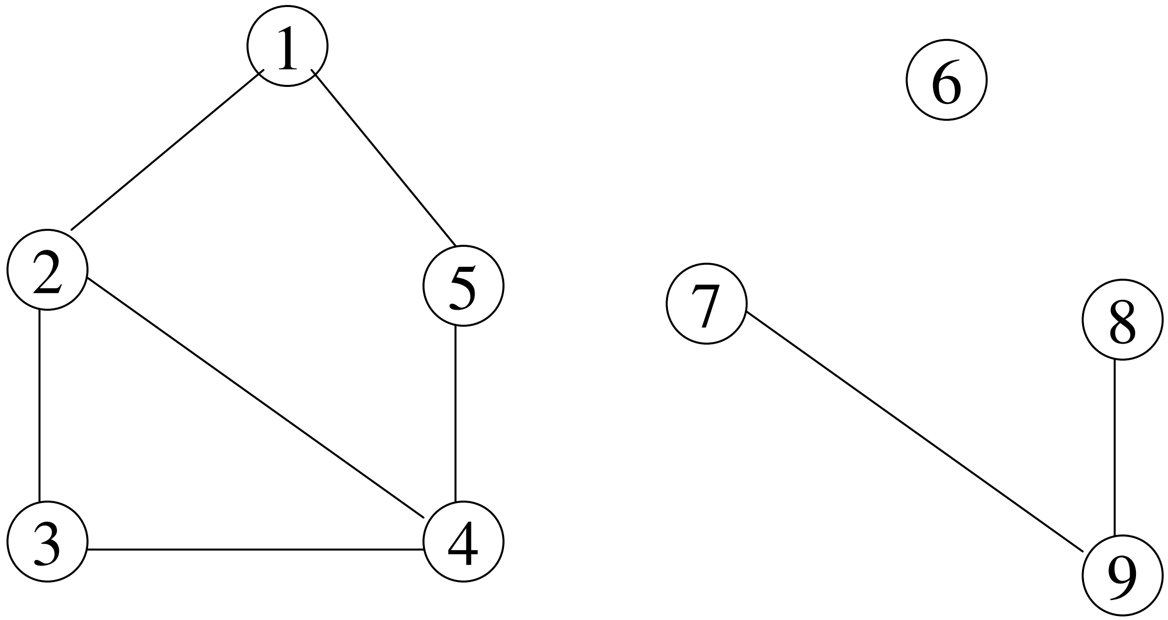
Connectedness in Graphs

- An undirected graph is ***connected*** if there is a path from every vertex to every other vertex.
- A directed graph is ***strongly connected*** if there is a path from every vertex to every other vertex.
- A directed graph is ***weakly connected*** if there would be a path from every vertex to every other vertex, disregarding the direction of the edges.
- A ***complete*** graph is one in which there is an edge between every pair of vertices.
- A ***connected component*** of a graph is any maximal connected subgraph. Connected components are sometimes simply called ***components***.

Disjoint Sets and Graphs

- Disjoint sets can be used to determine connected components of an undirected graph.
- For each edge, place its two vertices (u and v) in the same set -- i.e. union(u, v)
- When all edges have been examined, the forest of sets will represent the connected components.
- Two vertices, x, y, are connected if and only if find(x) = find(y)

Undirected Graph/Disjoint Set Example



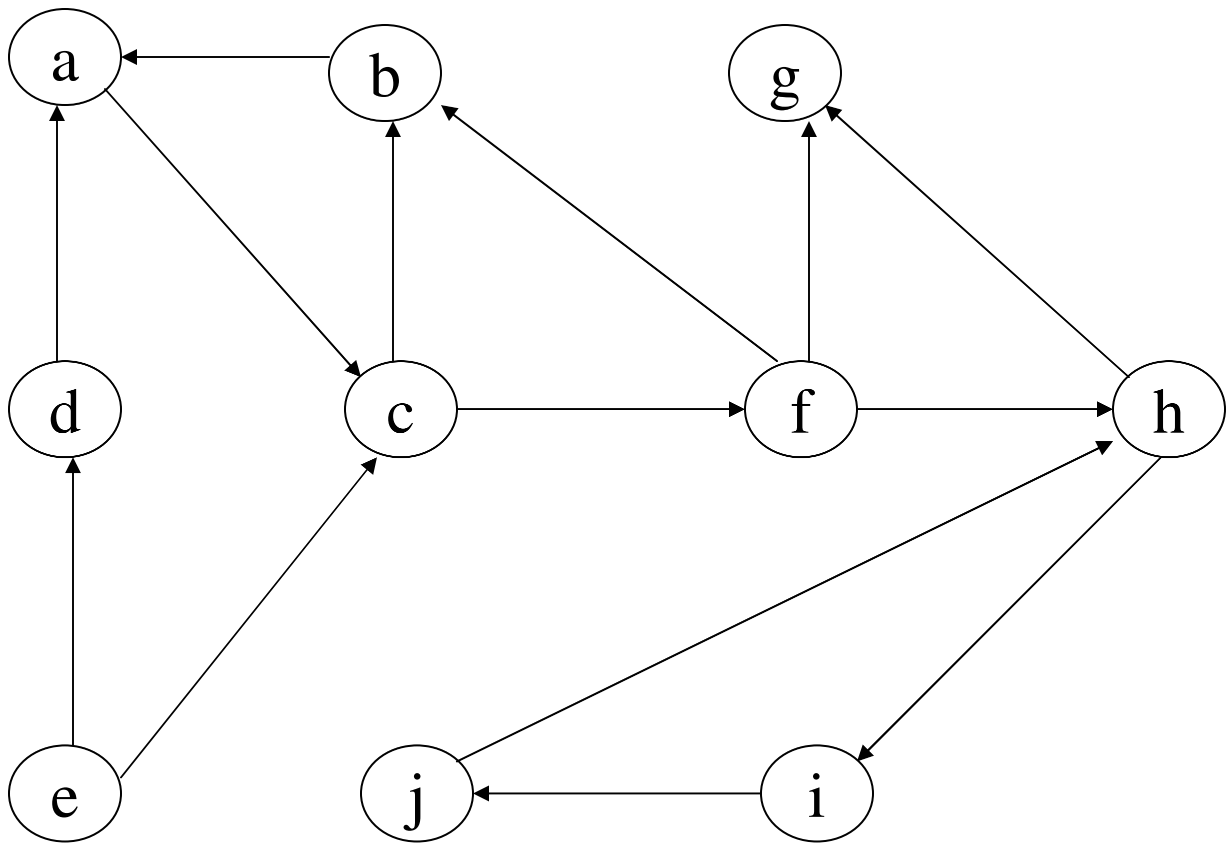
Sets representing connected components

{ 1, 2, 3, 4, 5 }

{ 6 }

{ 7, 8, 9 }

DiGraph / Strongly Connected Components



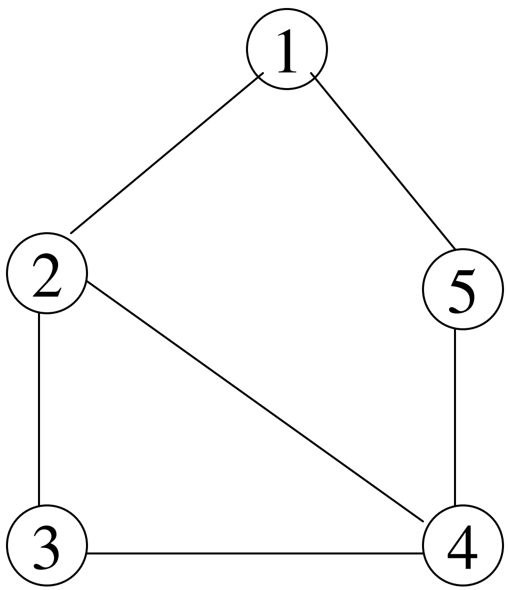
A Graph ADT

- Has some data elements
  - Vertices and Edges
- Has some operations
  - getDegree(u) -- Returns the degree of vertex u (outdegree of vertex u in directed graph)
  - getAdjacent(u) -- Returns a list of the vertices **<u>adjacent to</u>** vertex u (list of vertices that u points to for a directed graph)
  - isAdjacentTo(u, v) -- Returns TRUE if vertex v is adjacent to vertex u, FALSE otherwise.
- Has some associated algorithms to be discussed.

Adjacency Matrix Implementation

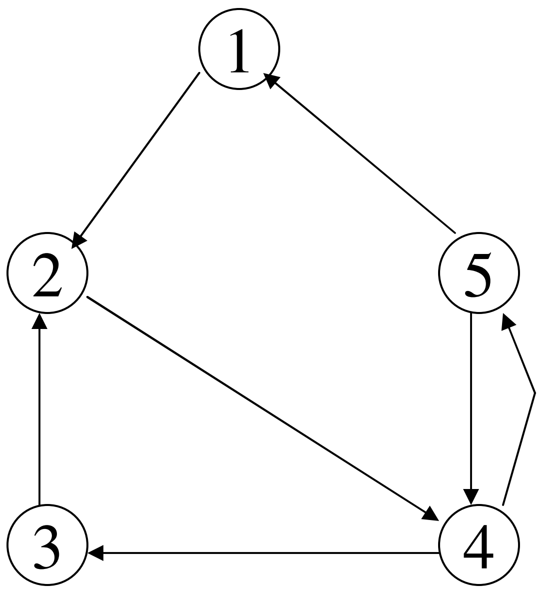
- Uses array of size $|V| \times |V|$ where each entry (i ,j) is boolean
  - TRUE if there is an edge from vertex i to vertex j
  - FALSE otherwise
  - store weights when edges are weighted
- Very simple, but large space requirement = $O(|V|^2)$
- Appropriate if the graph is dense.
- Otherwise, most of the entries in the table are FALSE.
- For example, if a graph is used to represent a street map like Manhattan in which most streets run E/W or N/S, each intersection is attached to only 4 streets and $|E| < 4*|V|$. If there are 3000 intersections, the table has 9,000,000 entries of which only 12,000 are TRUE.

Undirected Graph / Adjacency Matrix



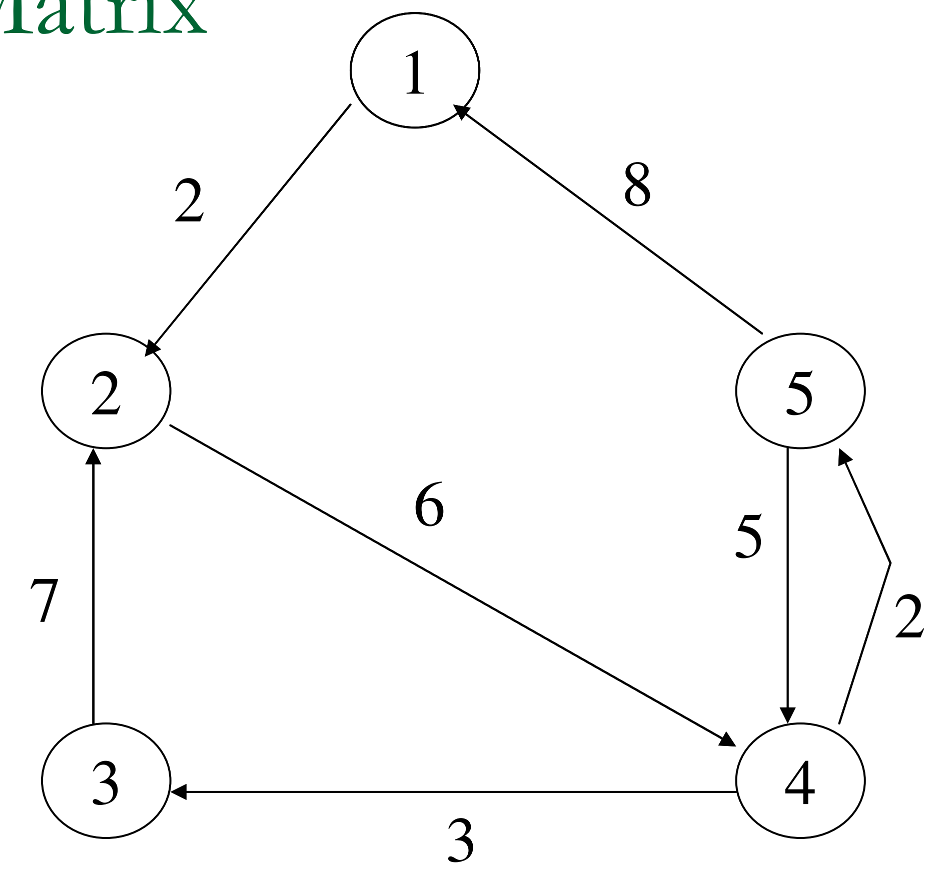
	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	0
3	0	1	0	1	0
4	0	1	1	0	1
5	1	0	0	1	0

Directed Graph / Adjacency Matrix



	1	2	3	4	5
1	0	1	0	0	0
2	0	0	0	1	0
3	0	1	0	0	0
4	0	0	1	0	1
5	1	0	0	1	0

Weighted, Directed Graph / Adjacency Matrix



	1	**2**	**3**	**4**	**5**
1	0	2	0	0	0
2	0	0	0	6	0
3	0	7	0	0	0
4	0	0	3	0	2
5	8	0	0	5	0

Adjacency Matrix Performance

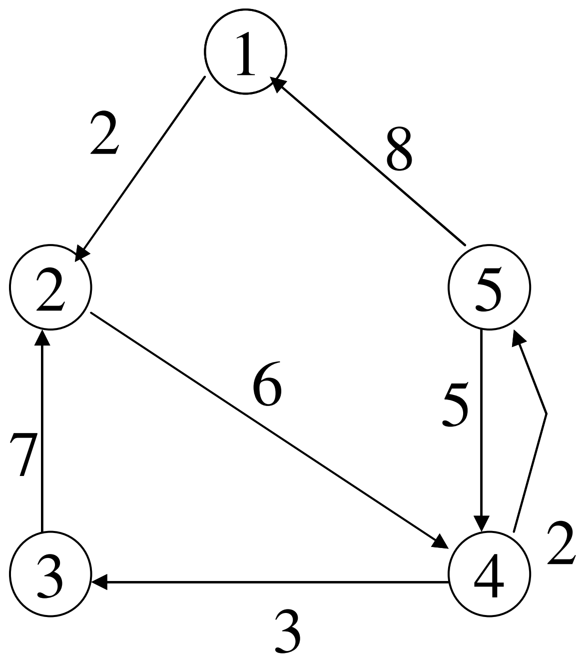
- Storage requirement:
  $O(|V|^2)$
- Performance:

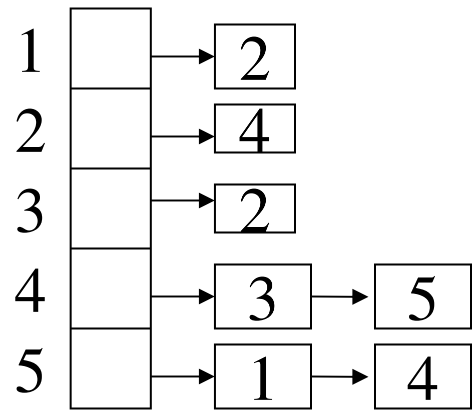
getDegree (u)	
isAdjacentTo(u, v)	
getAdjacent(u)	

Adjacency List Implementation

- If the graph is sparse, then keeping a list of adjacent vertices for each vertex saves space. Adjacency Lists are the commonly used representation. The lists may be stored in a data structure or in the Vertex object itself.
  - **Vector of lists**: A vector of lists of vertices. The i-th element of the vector is a list, $L_{i,}$ of the vertices adjacent to v_i.
- If the graph is sparse, then the space requirement is O(|E| + |V|), “linear in the size of the graph”
- If the graph is dense, then the space requirement is $O(|V|^2)$

Vector of Lists





Adjacency List Performance

- Storage requirement:
- Performance:

getDegree(u)	
isAdjacentTo(u, v)	
getAdjacent(u)	

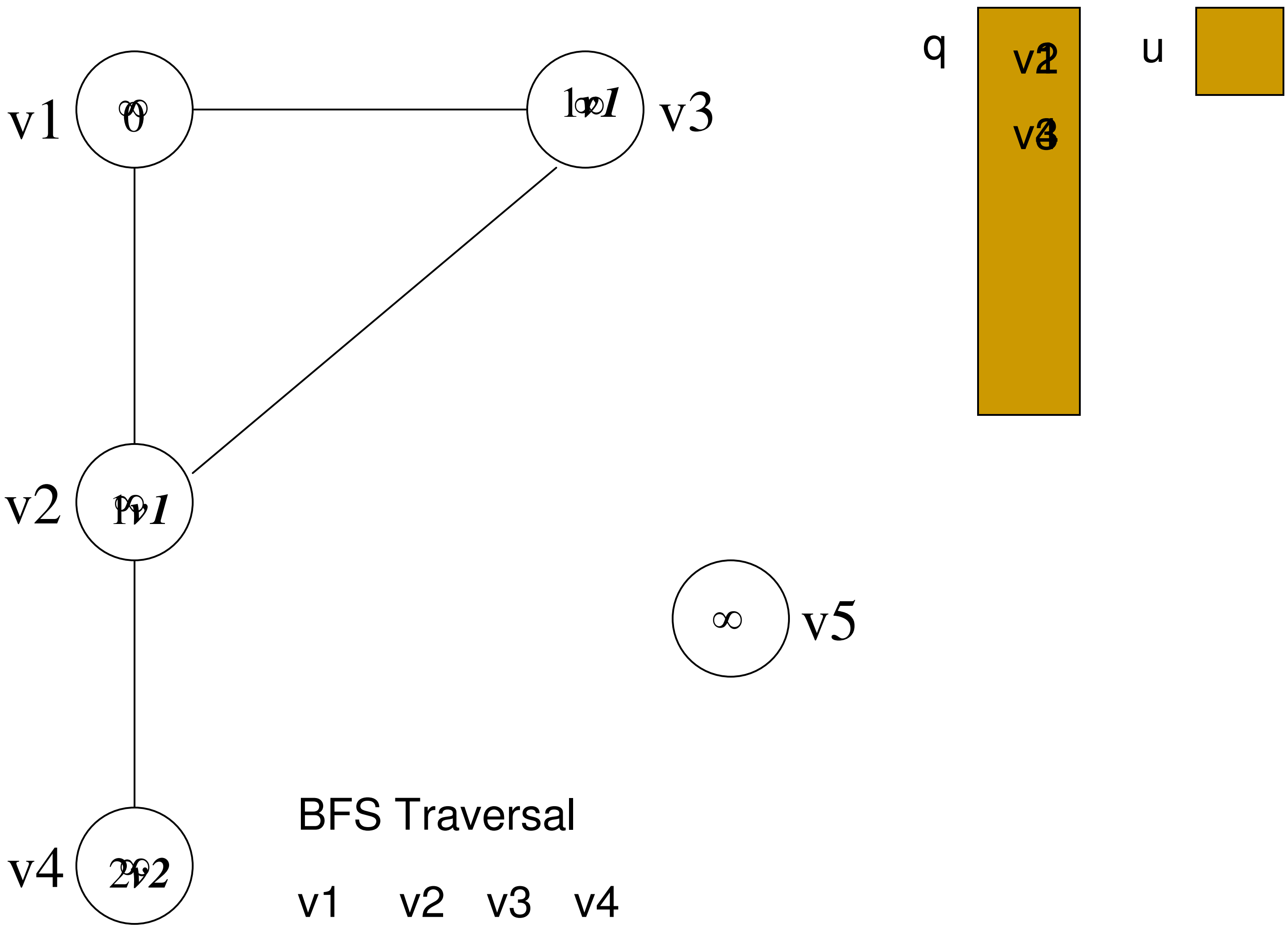
Graph Traversals

- Like trees, graphs can be traversed breadth-first or depth-first.
  - Use stack (or recursion) for depth-first traversal
  - Use queue for breadth-first traversal
- Unlike trees, we need to specifically guard against repeating a path from a cycle. Mark each vertex as "visited" when we encounter it and do not consider visited vertices more than once.

Breadth-First Traversal

```
void bfs()
{
    Queue<Vertex> q;
    Vertex u, w;

    for all v in V, d[v] = ∞          // mark each vertex unvisited
    q.enqueue(startvertex);           // start with any vertex
    d[startvertex] = 0;               // mark visited
    while ( !q.isEmpty() ) {
        u = q.dequeue( );
        for each Vertex w adjacent to u {
                if (d[w] == ∞) {      // w not marked as visited
                        d[w] = d[u]+1; // mark visited
                        path[w] = u;   // where we came from
                        q.enqueue(w);
                }
        }
    }
}
```

Breadth-First Example



BFS Traversal

v1 v2 v3 v4

Unweighted Shortest Path Problem

- Unweighted shortest-path problem: Given as input an unweighted graph, G = (V, E), and a distinguished starting vertex, s, find the shortest unweighted path from s to every other vertex in G.
- After running BFS algorithm with s as starting vertex, the length of the shortest path length from s to i is given by d[i]. If $d[i] = \infty$, then there is no path from s to i. The path from s to i is given by traversing path[] backwards from i back to s.

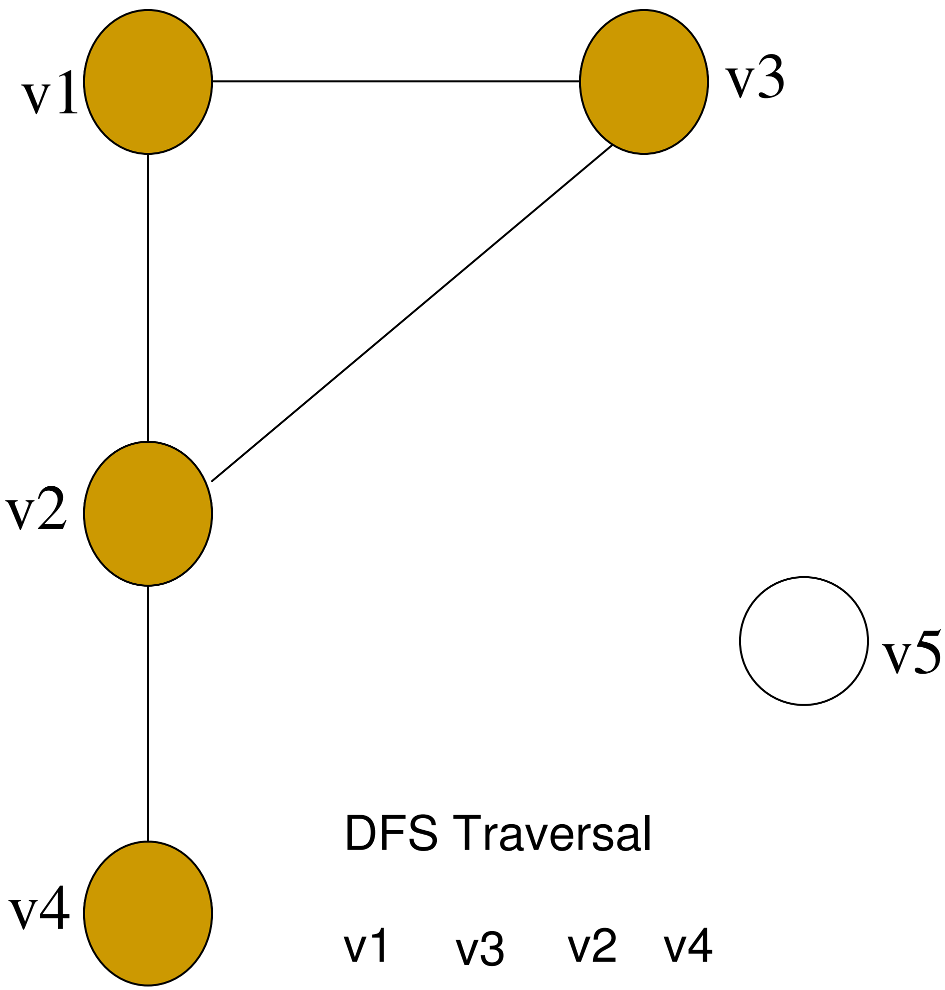
Recursive Depth First Traversal

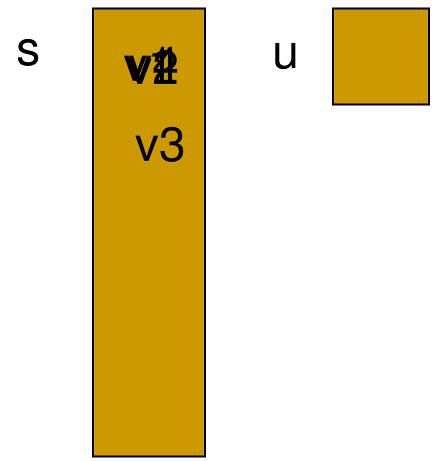
```
void dfs() {
   for (each v ∈ V)
        dfs(v)
}

void dfs(Vertex v)
{
   if (!v.visited)
   {
        v.visited = true;
        for each Vertex w adjacent to v)
               if ( !w.visited )
                      dfs(w)
   }
}
```

DFS with explicit stack

```
void dfs()
{
   Stack<Vertex> s;
   Vertex u, w;
   s.push(startvertex);
   startvertex.visited = true;
   while ( !s.isEmpty() ) {
        u = s.pop();
        for each Vertex w adjacent to u {
                if (!w.visited) {
                        w.visited = true;
                        s.push(w);
        }
   }
}
```

DFS Example





DFS Traversal

v1 v3 v2 v4

Traversal Performance

- What is the performance of DF and BF traversal?
- Each vertex appears in the stack or queue exactly once in the worst case. Therefore, the traversals are at least O(|V|). However, at each vertex, we must find the adjacent vertices. Therefore, df- and bf-traversal performance depends on the performance of the `getAdjacent` operation.

GetAdjacent

- Method 1: Look at every vertex (except u), asking “are you adjacent to u?”

```
List<Vertex> L;
for each Vertex v except u
  if (v.isAdjacentTo(u))
    L.push_back(v);
```

- Assuming O(1) performance for `push_back` and `isAdjacentTo`, then `getAdjacent` has O(|V|) performance and traversal performance is O($|V^2|$);

GetAdjacent (2)

- Method 2: Look only at the edges which impinge on u. Therefore, at each vertex, the number of vertices to be looked at is D(u), the degree of the vertex
- This approach is O(D(u)). The traversal performance is

  $$O\left(\sum_{i=1}^{|V|} D(v_i)\right) = \quad O\,(\,|E|\,)$$

  since `getAdjacent` is done O(|V|) times.
- However, in a disconnected graph, we must still look at every vertex, so the performance is O(|V| + |E|).

Number of Edges

- Theorem: The number of edges in an undirected graph $G = (V,E)$ is $O(|V|^2)$
- Proof: Suppose G is fully connected. Let $p = |V|$.
- Then we have the following situation:

vertex	connected to
1	2,3,4,5,…, p
2	1,3,4,5,…, p
…	
p	1,2,3,4,…,p-1

  - There are $p(p-1)/2 = O(|V|^2)$ edges.
- So $O(|E|) = O(|V|^2)$.

Weighted Shortest Path Problem

Single-source shortest-path problem:

Given as input a weighted graph, G = (V, E), and a distinguished starting vertex, s, find the shortest weighted path from s to every other vertex in G.

Use Dijkstra's algorithm

- Keep tentative distance for each vertex giving shortest path length using vertices visited so far.
- Record vertex visited before this vertex (to allow printing of path).
- At each step choose the vertex with smallest distance among the unvisited vertices (greedy algorithm).

Dijkstra's Algorithm

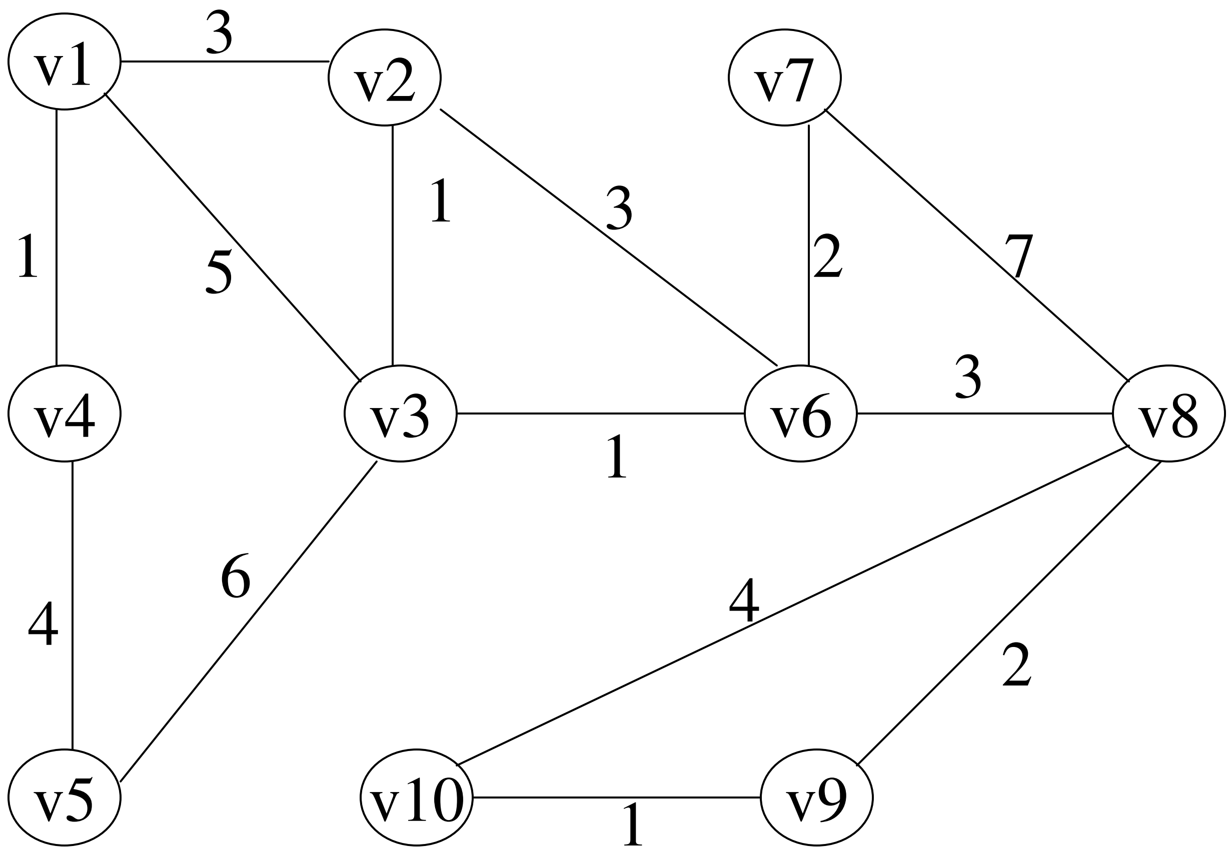
- The pseudo code for Dijkstra's algorithm assumes the following structure for a Vertex object

```
class Vertex
{
  public List adj;        //Adjacency list
  public boolean known;
  public DisType dist;  //DistType is probably int
  public Vertex path;
  //Other fields and methods as needed
}
```

Dijkstra’s Algorithm

```
void dijksra(Vertex start)
{
   for each Vertex v in V {
        v.dist = Integer.MAX_VALUE;
        v.known = false;
        v.path = null;
   }

   start.distance = 0;

   while there are unknown vertices {
        v = unknown vertex with smallest distance
        v.known = true;
        for each Vertex w adjacent to v
               if (!w.known)
                      if (v.dist + weight(v, w)< w.distance){
                             decrease(w.dist to v.dist+weight(v, w))
                             w.path = v;
                      }
   }
}
```

Dijkstra Example



Correctness of Dijkstra's Algorithm

- The algorithm is correct because of a property of shortest paths:
- If $P_k = v_1, v_2, ..., v_j, v_k$, is a shortest path from v_1 to v_k, then $P_j = v_1, v_2, ..., v_j$, must be a shortest path from v_1 to v_j. Otherwise P_k would not be as short as possible since P_k extends P_j by just one edge (from v_j to v_k)
- Also, P_j must be shorter than P_k (assuming that all edges have positive weights). So the algorithm must have found P_j on an earlier iteration than when it found P_k.
- i.e. Shortest paths can be found by extending earlier known shortest paths by single edges, which is what the algorithm does.

Running Time of Dijkstra's Algorithm

- The running time depends on how the vertices are manipulated.
- The main 'while' loop runs O(|V|) time (once per vertex)
- Finding the "unknown vertex with smallest distance" (inside the while loop) can be a simple linear scan of the vertices and so is also O(|V|). With this method the total running time is $O(|V|^2)$. This is acceptable (and perhaps optimal) if the graph is dense ($|E| = O(|V|^2)$) since it runs in linear time on the number of edges.
- If the graph is sparse, ($|E| = O(|V|)$), we can use a priority queue to select the unknown vertex with smallest distance, using the deleteMin operation (O(lg |V|)). We must also decrease the path lengths of some unknown vertices, which is also O(lg|V|). The deleteMin operation is performed for every vertex, and the "decrease path length" is performed for every edge, so the running time is
  $O(|E| \lg|V| + |V|\lg|V|) = O((|V|+|E|) \lg|V|) = O(|E| \lg|V|)$ if all vertices are reachable from the starting vertex

Dijkstra and Negative Edges

- Note in the previous discussion, we made the assumption that all edges have positive weight. If any edge has a negative weight, then Dijkstra's algorithm fails. Why is this so?
- Suppose a vertex, u, is marked as "known". This means that the shortest path from the starting vertex, s, to u has been found.
- However, it's possible that there is negatively weighted edge from an unknown vertex, v, back to u. In that case, taking the path from s to v to u is actually shorter than the path from s to u without going through v.
- Other algorithms exist that handle edges with negative weights for weighted shortest-path problem.

Directed Acyclic Graphs

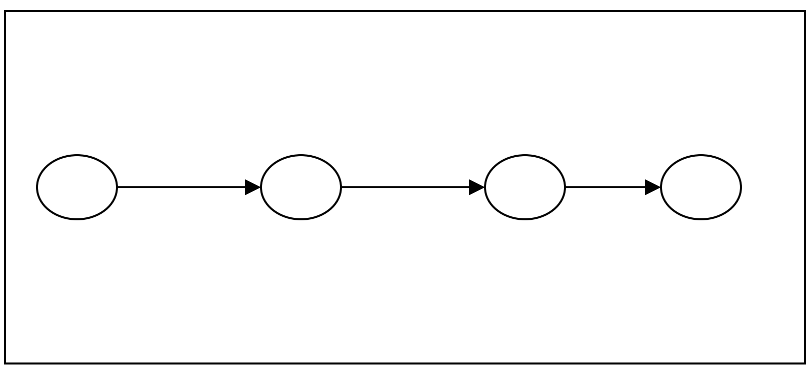
- A ***directed acyclic graph*** is a directed graph with no cycles.
- A ***strict partial order*** R on a set S is a binary relation such that
  - for all a∈ S, aRa is false (irreflexive property)
  - for all a,b,c ∈ S, if aRb and bRc then aRc is true (transitive property)
- To represent a partial order with a DAG:
  - represent each member of S as a vertex
  - for each pair of vertices (a,b), insert an edge from a to b if and only if aRb

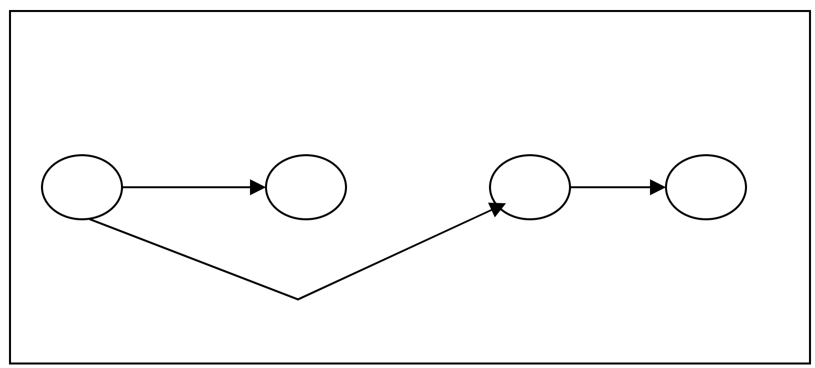
More Definitions

- Vertex i is a ***predecessor*** of vertex j if and only if there is a path from i to j.
- Vertex i is an ***immediate predecessor*** of vertex j if and only if (i, j) is an edge in the graph.
- Vertex j is a ***successor*** of vertex i if and only if there is a path from i to j.
- Vertex j is an ***immediate successor*** of vertex i if and only if (i, j) is an edge in the graph.
- The ***indegree*** of a vertex, v, is the number of edges (u, v), i.e. the number of edges that come "into" v.

Topological Ordering

- A topological ordering of the vertices of a DAG G = (V,E) is a linear ordering such that, for vertices i, j $\in$ V, if i is a predecessor of j, then i precedes j in the linear order, i.e. if there is a path from v_i to v_j, then v_i comes before v_j in the linear order

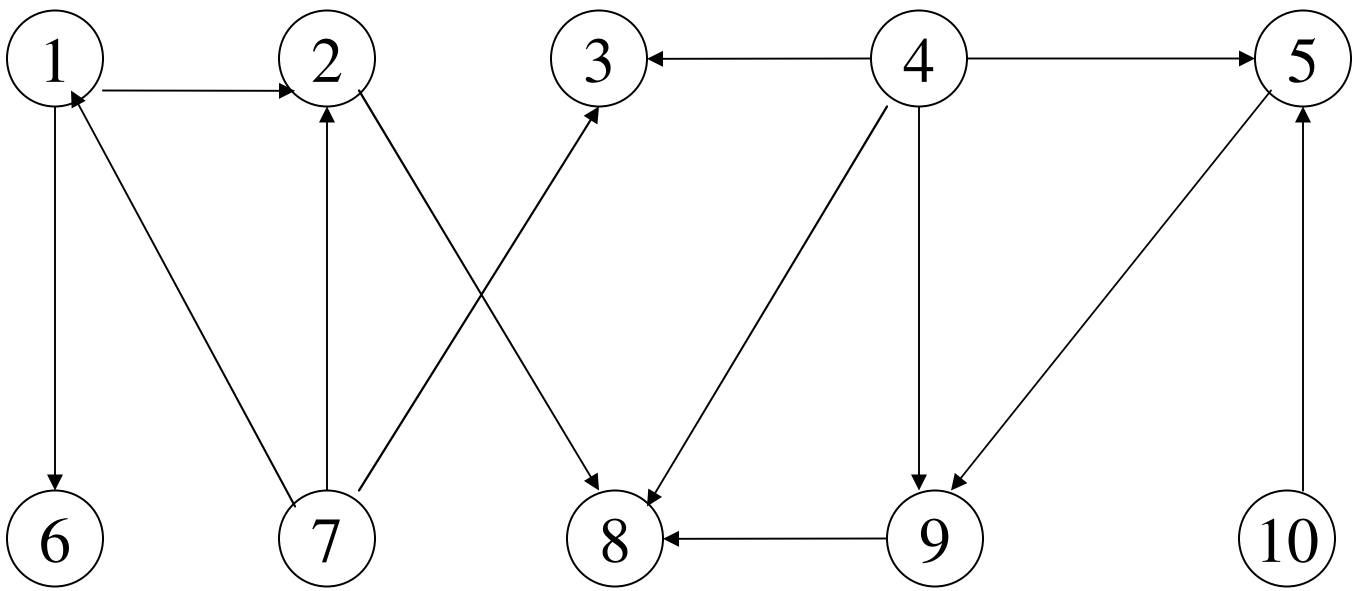




Topological Sort

```
void topsort( ) throws CycleFoundException
{
    Queue<Vertex> q = new Queue<Vertex>( );
    int counter = 0;

    for each Vertex v
        if( v.indegree == 0 )
            q.enqueue( v );

    while( !q.isEmpty( ) )
    {
        Vertex v = q.dequeue( );
        v.topNum = ++counter;  // Assign next number

        for each Vertex w adjacent to v
            if( --w.indegree == 0 )
                q.enqueue( w );
    }

    if( counter != NUM_VERTICES )
        throw new CycleFoundException( );
}
```

TopSort Example



Running Time of TopSort

1. At most, each vertex is enqueued just once, so there are O(|V|) constant time queue operations.
2. The body of the for loop is executed at most once per edges = O(|E|)
3. The initialization is proportional to the size of the graph if adjacency lists are used = O(|E| + |V|)
4. The total running time is therefore O (|E| + |V|)