Tree ADT

Tree definition

– A tree is a set of nodes.
– The set may be empty
– If not empty, then there is a distinguished node \( r \), called root and zero or more non-empty subtrees \( T_1, T_2, \ldots, T_k \), each of whose roots are connected by a directed edge from \( r \).

Basic Terminology

– Root of a subtree is a child of \( r \). \( r \) is the parent.
– All children of a given node are called siblings.
– A leaf (or external) node has no children.
– An internal node is a node with one or more children
More Tree Terminology

A path from node $V_i$ to node $V_k$ is a sequence of nodes such that $V_i$ is the parent of $V_{i+1}$ for $1 \leq i \leq k$.
The length of this path is the number of edges encountered. The length of the path is one less than the number of nodes on the path ($k - 1$ in this example).
The depth of any node in a tree is the length of the path from root to the node.
All nodes of the same depth are at the same level.
The depth of a tree is the depth of its deepest leaf.
The height of any node in a tree is the length of the longest path from the node to a leaf.
The height of a tree is the height of its root.
If there is a path from $V_1$ to $V_2$, then $V_1$ is an ancestor of $V_2$ and $V_2$ is a descendant of $V_1$. 

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Tree Storage

A tree node contains:

- Element
- Links
  - to each child
  - to sibling and first child
Binary Trees

A *binary tree* is a rooted tree in which no node can have more than two children AND the children are distinguished as *left* and *right*.

A *full BT* is a BT in which every node either has two children or is a leaf (every interior node has two children).
FBT Theorem

Theorem: A FBT with $n$ internal nodes has $n + 1$ leaf nodes.

Proof by strong induction on the number of internal nodes, $n$:

Base case: BT of one node (the root) has:
- zero internal nodes
- one external node (the root)

Inductive Assumption:
Assume all FBTs with up to and including $n$ internal nodes have $n + 1$ external nodes.
Proof (cont)

Inductive Step (prove true for tree with $n + 1$ internal nodes)
(i.e a tree with $n + 1$ internal nodes has $(n + 1) + 1 = n + 2$ leaves)

- Let $T$ be a FBT of $n$ internal nodes.
- It therefore has $n + 1$ external nodes (Inductive Assumption)
- Enlarge $T$ so it has $n+1$ internal nodes by adding two nodes to some leaf. These new nodes are therefore leaf nodes.
- Number of leaf nodes increases by 2, but the former leaf becomes internal.
- So,
  - # internal nodes becomes $n + 1$,
  - # leaves becomes $(n + 1) + 1 = n + 2$
Proof (more rigorous)

Inductive Step (prove for n+1):

– Let T be any FBT with n + 1 internal nodes.
– Pick any leaf node of T, remove it and its sibling.
– Call the resulting tree T1, which is a FBT
– One of the internal nodes in T is changed to a external node in T1
  • T has one more internal node than T1
  • T has one more external node than T1
– T1 has n internal nodes and n + 1 external nodes (by inductive assumption)
  • Therefore T has (n + 1) + 1 external nodes.
Perfect Binary Tree

A *perfect BT* is a full BT in which all leaves have the same depth.
PBT Theorem

Theorem: The number of nodes in a PBT is $2^{h+1} - 1$, where $h$ is height.

Proof by strong induction on $h$, the height of the PBT:

Notice that the number of nodes at each level is $2^l$.

(Proof of this is a simple induction - left to student as exercise). Recall that the height of the root is 0.

Base Case:

The tree has one node; then $h = 0$ and $n = 1$.

and $2^{(h + 1)} = 2^{(0 + 1)} - 1 = 2^1 - 1 = 2 - 1 = 1 = n$
Proof of PBT Theorem (cont)

Inductive Assumption:
Assume true for all trees with height \( h \leq H \)

Prove true for tree with height \( H+1 \):
Consider a PBT with height \( H + 1 \). It consists of a root and two subtrees of height \( H \). Therefore, since the theorem is true for the subtrees (by the inductive assumption since they have height = \( H \))

\[
\begin{align*}
 n &= (2^{(H+1)} - 1) \quad \text{for the left subtree} \\
 &\quad + (2^{(H+1)} - 1) \quad \text{for the right subtree} \\
 &\quad + 1 \quad \text{for the root} \\
 &= 2 \times (2^{(H+1)} - 1) + 1 \\
 &= 2^{((H+1)+1)} - 2 + 1 = 2^{((H+1)+1)} - 1. \quad \text{QED}
\end{align*}
\]
Other Binary Trees

Complete Binary Tree

A complete BT is a perfect BT except that the lowest level may not be full. If not, it is filled from left to right.

Augmented Binary Tree

An augmented binary tree is a BT in which every unoccupied child position is filled by an additional “augmenting” node.
Path Lengths

The *internal path length* (IPL) of a rooted tree is the sum of the depths of all of its internal nodes.

The *external path length* (EPL) of a rooted tree is the sum of the depths of all the external nodes.

There is a relationship between the IPL and EPL of Full Binary Trees.

If $n_i$ is the number of internal nodes in a FBT, then

$$EPL(n_i) = IPL(n_i) + 2n_i$$

Example:

- $n_i = \ldots$
- $EPL(n_i) = \ldots$
- $IPL(n_i) = \ldots$
- $2 \cdot n_i = \ldots$
Proof of Path Lengths

Prove: $EPL(n_i) = IPL(n_i) + 2 \times n_i$ by induction on number of internal nodes

Base: $n_i = 0$ (single node, the root)

- $EPL(n_i) = 0$
- $IPL(n_i) = 0$; $2 \times n_i = 0$; $0 = 0 + 0$

IH: Assume true for all FBT with $n_i < N$

Prove for $n_i = N$. 
Proof: Let $T$ be a FBT with $n_i = N$ internal nodes.

Let $n_{iL}$, $n_{iR}$ be # of internal nodes in L, R subtrees of $T$
then $N = n_i = n_{iL} + n_{iR} + 1 \Rightarrow n_{iL} < N; n_{iR} < N$

So by IH:

$$EPL(n_{iL}) = IPL(n_{iL}) + 2 \ n_{iL}$$
and
$$EPL(n_{iR}) = IPL(n_{iR}) + 2 \ n_{iR}$$

For $T$,

$$EPL(n_i) = EPL(n_{iL}) + n_{iL} + 1 + EPL(n_{iR}) + n_{iR} + 1$$

By substitution

$$EPL(n_i) = IPL(n_{iL}) + 2 \ n_{iL} + n_{iL} + 1 + IPL(n_{iR}) + 2 \ n_{iR} + n_{iR} + 1$$

Notice that $IPL(n_i) = IPL(n_{iL}) + IPL(n_{iR}) + n_{iL} + n_{iR}$

By combining terms

$$EPL(n_i) = IPL(n_i) + 2 \ (n_{iR} + n_{iL} + 1)$$

But $n_{iR} + n_{iL} + 1 = n_i$, therefore

$$EPL(n_i) = IPL(n_i) + 2 \ n_i$$

QED

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Constructing Trees

Is it possible to reconstruct a BT from just one of its pre-order, inorder, or post-order sequences?
Constructing Trees (cont)

Given two sequences (say pre-order and inorder) is the tree unique?
Tree Implementations

What should methods of a tree class be?
Tree class

template <class Object>
class Tree {
  public:
    Tree(const Object& notFnd);
    Tree (const Tree& rhs);
    ~Tree();

    const Object &find(const Object& x) const;
    bool isEmpty() const;
    void printTree() const;
    void makeEmpty();
    void insert (const Object& x);
    void remove (const Object& x);
    const Tree& operator=(const Tree &rhs);
}
Tree class (cont)

private:
    TreeNode<Object> *root;
const Object ITEM_NOT_FOUND;
const Object& elementAt(TreeNode<Object> *t) const;
void insert (const Object& x, TreeNode<Object> **& t) const;
void remove (const Object& x, TreeNode<Object> **& t) const;
TreeNode<Object> *find(const Object& x,
                       TreeNode<Object> * t) const;
void makeEmpty(TreeNode<Object> **& t) const;
void printTree(TreeNode<Object> * t) const;
TreeNode<Object> * clone(TreeNode<Object> * t) const;
};

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Tree Implementations

Fixed Binary
- element
- left pointer
- right pointer

Fixed K-ary
- element
- array of K child pointers

Linked Sibling/Child
- element
- firstChild pointer
- nextSibling pointer
TreeNode : Static Binary

template <class Object>
class BinaryNode {
  Object element;
  BinaryNode *left;
  BinaryNode *right;

  BinaryNode(const Object& theElement,
              BinaryNode* lt,
              BinaryNode* rt)
    : element (theElement), left(lt), right(rt) {}

  friend class Tree<Object>;
};
template <class Object>
BinaryNode<Object> *Tree<Object> ::
find(const Object& x, BinaryNode<Object> * t) const {
    BinaryNode<Object> *ptr;

    if (t == NULL)
        return NULL;
    else if (x == t->element)
        return t;
    else if (ptr = find(x, t->left))
        return ptr;
    else
        return(ptr = find(x, t->right));
}
Remove : Static Binary
TreeNode : Static K-ary

template <class Object>
class KaryNode {
    Object element;
    KaryNode * children[MAX_CHILDREN];

    KaryNode(const Object& theElement);

    friend class Tree<Object>;
};
Find : Static K-ary

template <class Object>
KaryNode<Object> *KaryTree<Object> ::
find(const Object& x, KaryNode<Object> *t) const
{
    KaryNode<Object> *ptr;

    if (t == NULL)
        return NULL;
    else if (x == t->element)
        return t;
    else {
        i = 0;
        while (((i < MAX_CHILDREN)
            && !(ptr = find(x, t->children[i]))) i++;
        return ptr;
    }
}

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Remove: Static K-ary
TreeNode : Sibling/Child

template <class Object>
class KTreeNode {
    Object element;
    KTreeNode *nextSibling;
    KTreeNode *firstChild;

    KTreeNode(const Object& theElement,
              KTreeNode *ns,
              KTreeNode *fc)
      : element (theElement),  nextSibling(ns),
        firstChild(fc) {} 

    friend class Tree<Object>;
};

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Find : Sibling/Child

template <class Object>
KTreeNode<Object> *Tree<Object> ::
find(const Object& x, KTreeNode<Object> *t) const
{
    KTreeNode<Object> *ptr;

    if (t == NULL)
        return NULL;
    else if (x == t->element)
        return t;
    else if (ptr = find(x, t->firstChild))
        return ptr;
    else
        return (ptr = find(x, t->nextSibling));
}