1. (10 points) Show the red-black trees after successively inserting the keys 41, 38, 31, 12, 19, 8 into and initially empty red-black tree.

Insert 41: 41

Insert 38: 41 38

Insert 31: 41 38 31

Insert 12: 41 38 31 12

**Problem 1:** 10/10  
**Problem 2:** 10/10  
**Problem 3:** 5/5
Grading Rubric:
Grade each insertion by itself. The first 2 insertions of 41, 38 are worth 1 point each.
Grade all other insertions being worth 2 points.

To keep grading simple each step has to have the correct tree after that insertion.
If they have messed up a step the following steps can not be right, so direct them to the solution.

There is a total of 10 points that come from this problem.
2. (10 points) Show the red-black trees that result from the successive deletion of the keys in the order 5, 25, 50, 75, 99.

Delete 5:

Done because 5 was a red leaf

Delete 25:

Since P was red we can change it to black and absorb that extra blackness.

Delete 50:

Done because V is red and we can absorb the extra blackness by changing the red node to black.

Delete 75:

Since P+ is the root of the tree that extra blackness will not affect the black height of the tree, and we are done.
Delete 99:

Splice

Result

apply

Case 1

V+ Absorb 100

Done because V is red and we can absorb the extra blackness by changing the red node to black.

Grading Rubric:

Each deletion is worth 2 points.

To keep grading simple each step has to have the correct tree after the deletion is completed. If they have messed up a step the following steps can not be right, so direct them to the solution.

There is a total of 10 points that come from this problem.
3. (5 points) Consider a red-black tree formed by inserting $n$ nodes into an initially empty red-black tree. Argue that if $n > 1$, the tree has at least one red node. Your argument must be in the form of a proof by induction. There are two cases you will need to take into consideration when making your inductuve step. The trivial case when Red Node $N + 1$ is inserted as a child of a black node does not need to be solved.

Proof by Induction:

Base Case: $N = 2$

Inductive Hypothesis: Assume that for a red-black tree of $n$ nodes, where $1 < n \leq N$, there exists at least 1 red node.

Inductive Step: Prove that for a red-black tree of $N + 1$ nodes there exists at least 1 red node.

There are 2 cases we have to of interest on the $N + 1$ th insertion.

Case 1: Red Node $N + 1$ is inserted as a child of a black node. This is the base case which we proved to be true already. This is the trivial case.

Case 2: Red Node $N + 1$ is inserted as a child of a red node. We must look at the insertion cases that have X as a child of a red node P.

Insertion Case 1: X remains red after the recoloring of P, G, U. It also remains the same color after we move reference X to Grandparent of X. Thus we have atleast 1 red node.

Insertion Case 2: The parent P of X remains red after the Zig-Zag rotation of X about P and then G. P remains red after recoloring X and G. Thus we have atlease 1 red node.

Insertion Case 3: X remains red after the rotation of P about G. X remains red after the recoloring of P and G. Thus we have atleast 1 red node.

Insertion Case 0: Not included since $n! = 1$, Insertion Cases 2 & 3 are terminating conditions, and Insertion Case 1 still produces 1 red node.

Q.E.D.

Please read the question before Grading.

5 points to a solid proof. Must show for all cases.
4 points to an attempt at a proof but has not proven all cases.
3 points to not attempt at a formal proof but still looks at some required cases.
0 points for no real attempt at the problem.