Binomial Queues

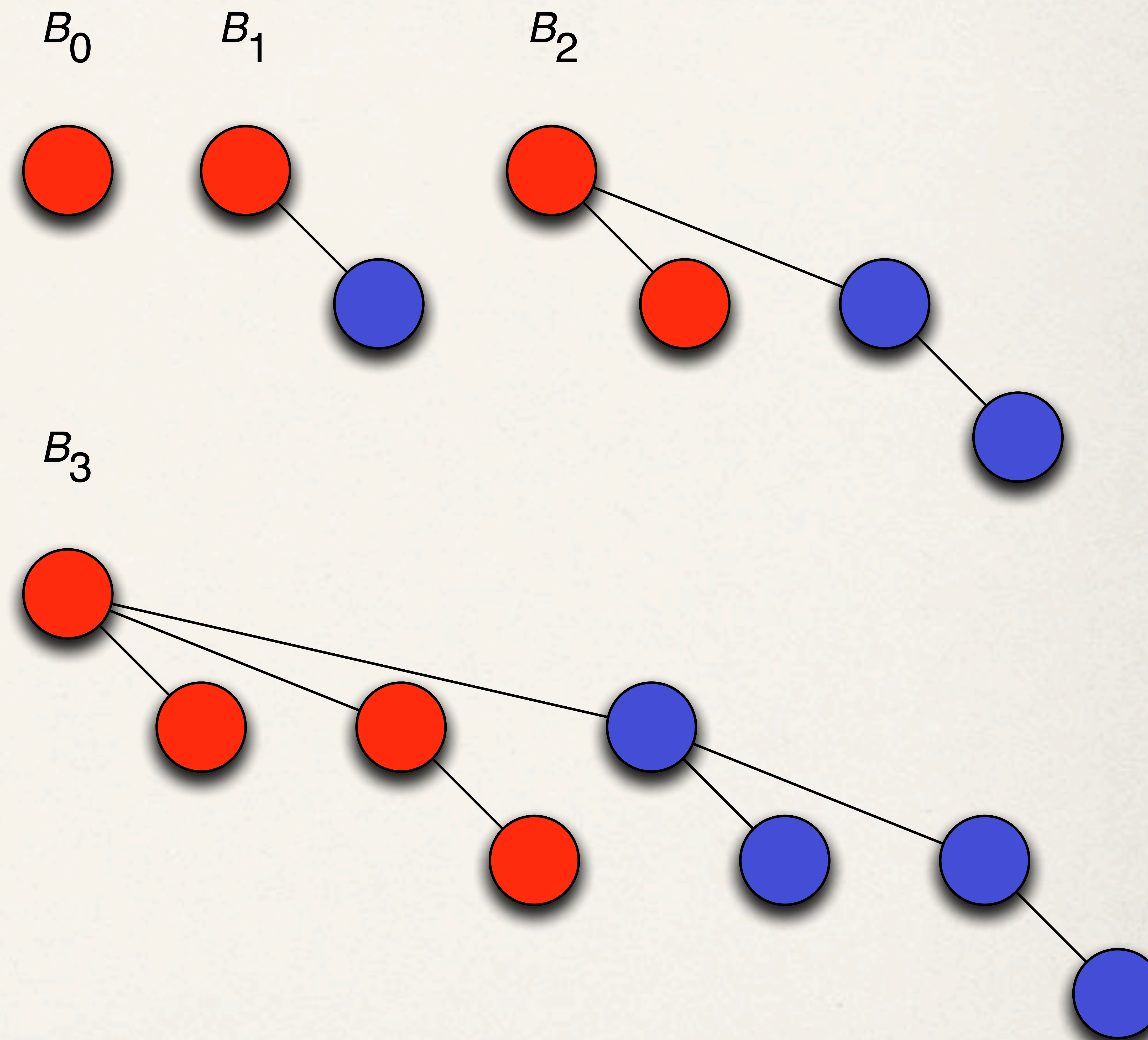
Section 6.8

Heap Operations: Merge

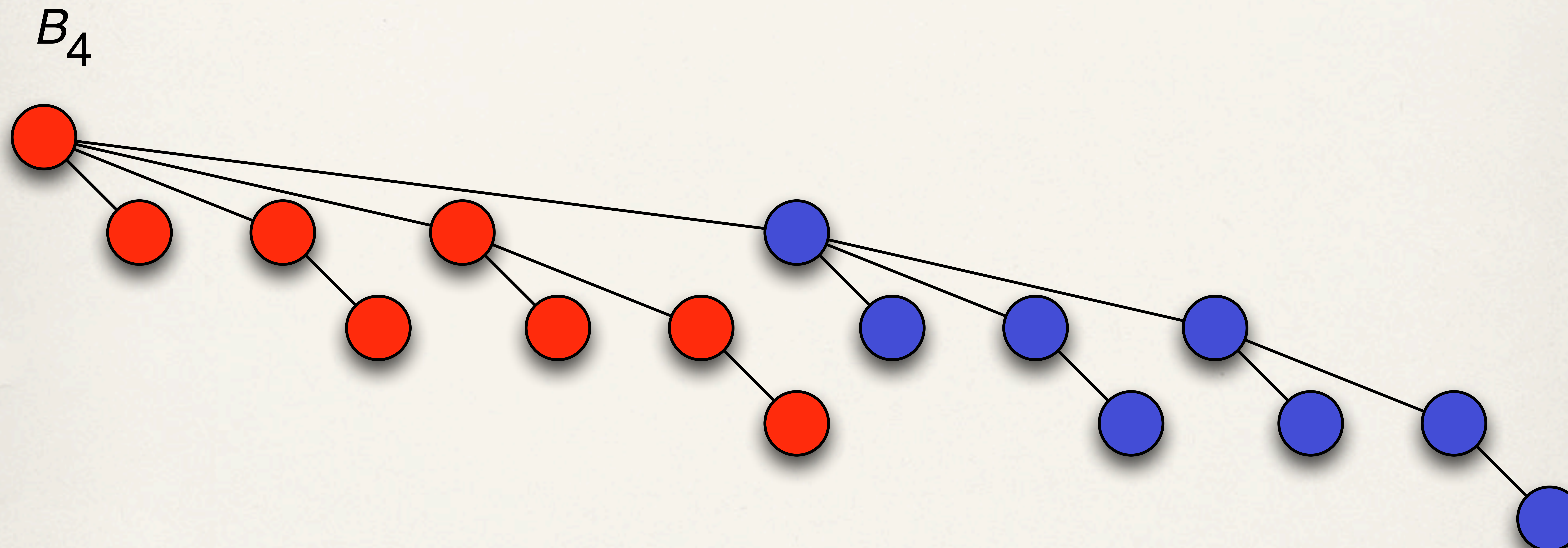
- Given two binary heaps H_1 and H_2, produce a new heap H' combining H_1 and H_2
  - Binary heaps take $\Theta(n_1 + n_2)$ time to merge
  - i.e. they can never merge in better than linear time
- We can do better, however
  - Merge in $O(\log N)$ time
  - this comes at the expensive of a slight performance hit on our other operations

Binomial Trees

- **Binomial trees** are recursive defined
  - Start with one node
    - This is a binomial tree of **height** 0
  - To form a tree of height *k*, attach two trees of height *k* - 1 together
    - Attach one as a child of the root of the other



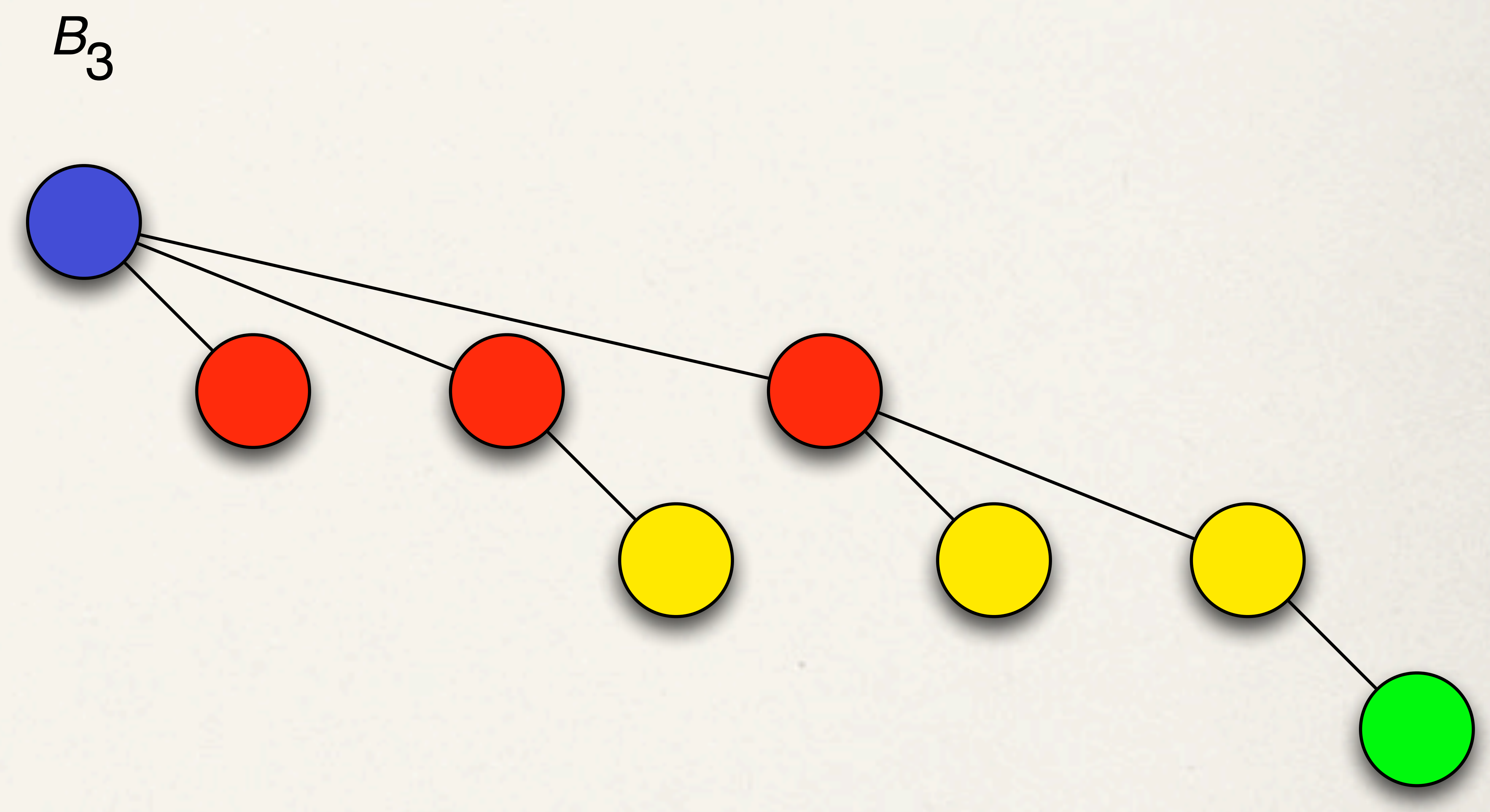
B_4



Binomial Tree Size

- A binomial tree of height k has 2^k nodes
    - Conversely, a binomial tree with n nodes has $\log_2(n)$ height
- The number of nodes at level d of a tree with height k is the binomial coefficient:

$$\binom{k}{d} = \frac{k!}{d!(k-d)!}$$

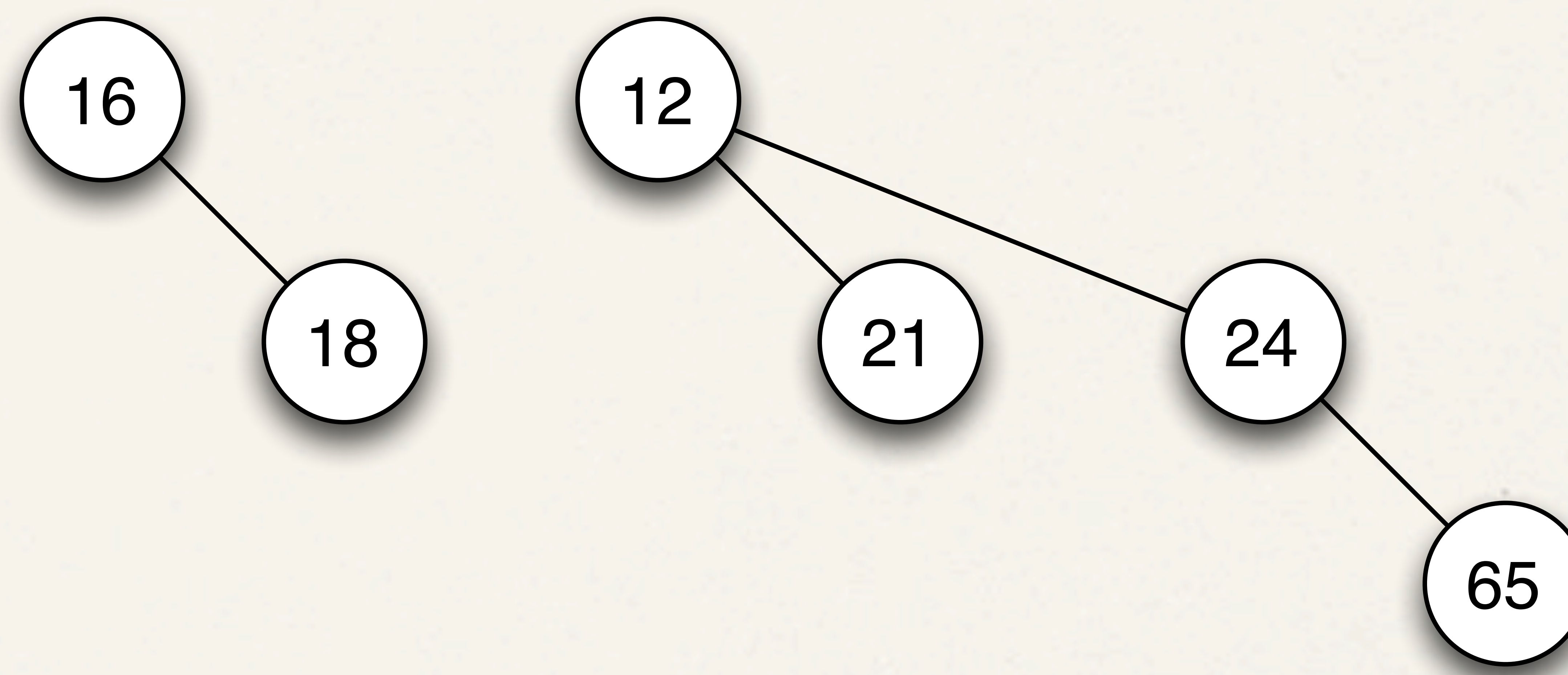


Binomial Queues

- **Binomial Heaps / Binomial Queues**
    - use a **forest** of binomial trees
        - use each binomial tree {0,1} times
    - impose heap ordering on each binomial tree
    - no relationship between the roots of each tree

Binomial Queues

H_1:

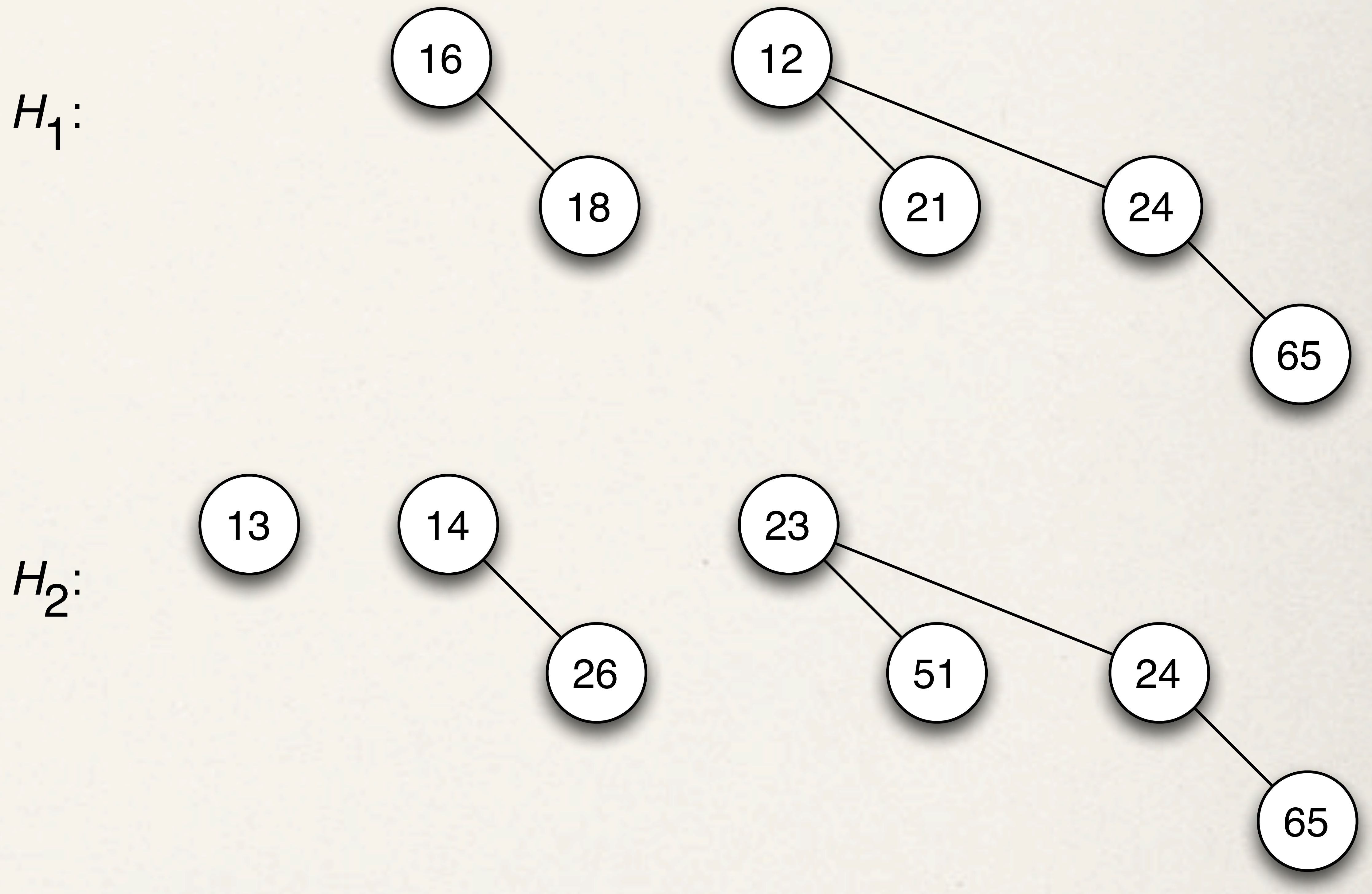


Binomial Queue Size

- A binomial queue H with N nodes has $O(\log N)$ binomial trees
  - let k be the largest integer such that $2^k \leq N$
  - observe that $k \leq \log_2(N)$
  - N can be written as the sum of unique powers of 2, the largest of which is 2^k
    - this sum uses each power of 2 {0,1} times
  - the sum has at most $k + 1$ terms in it
  - each term corresponds to a binomial tree of 2^n nodes in the forest of H

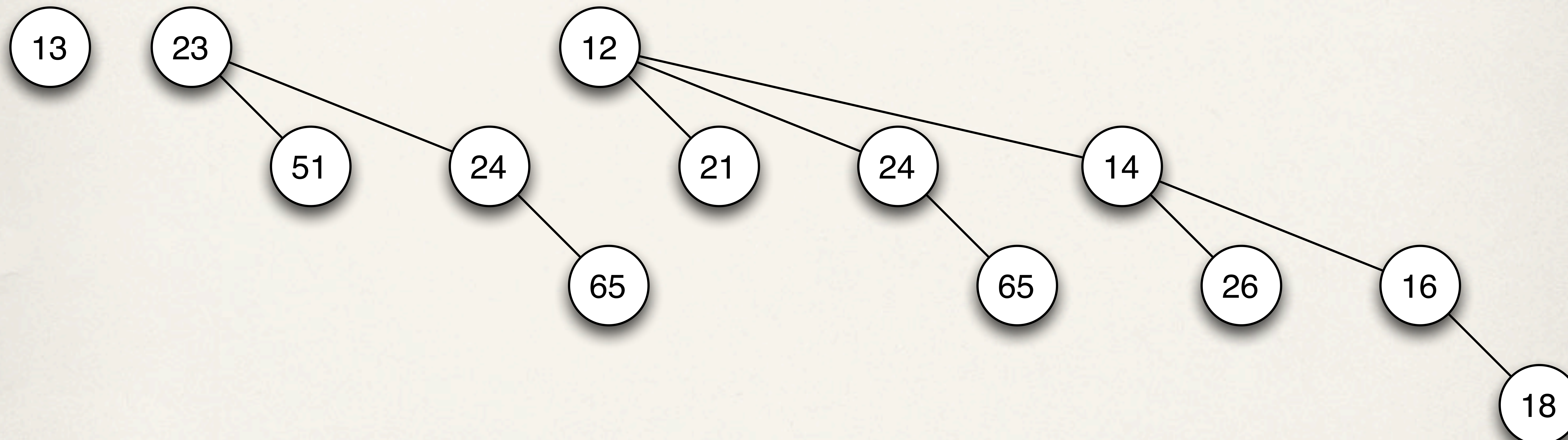
Merge

- "Add" corresponding trees from the two forests
- For k from 0 to maxheight
  - If neither queue has a B_k, skip
  - If only 1, leave it
  - If two, attach the larger priority root as a child of the other, producing a tree of height $k + 1$
  - If three, pick two to merge, leave 1



$O(\log N)$!

After Merging H_1 and H_2

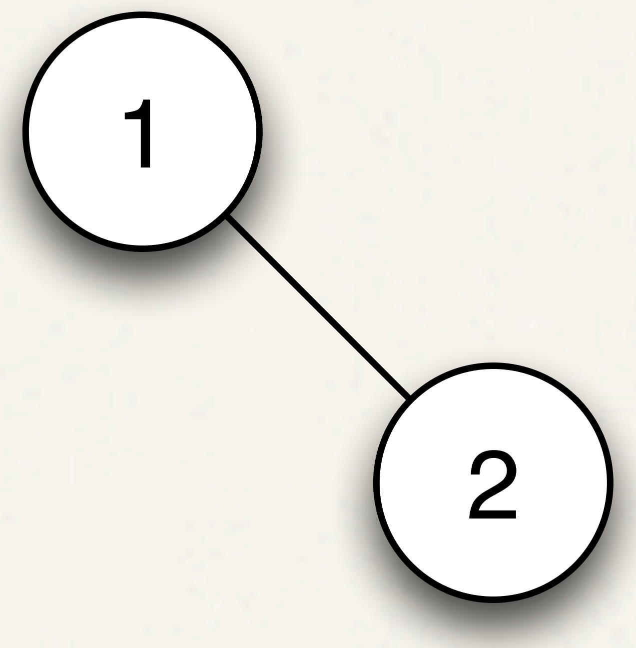


Insertion

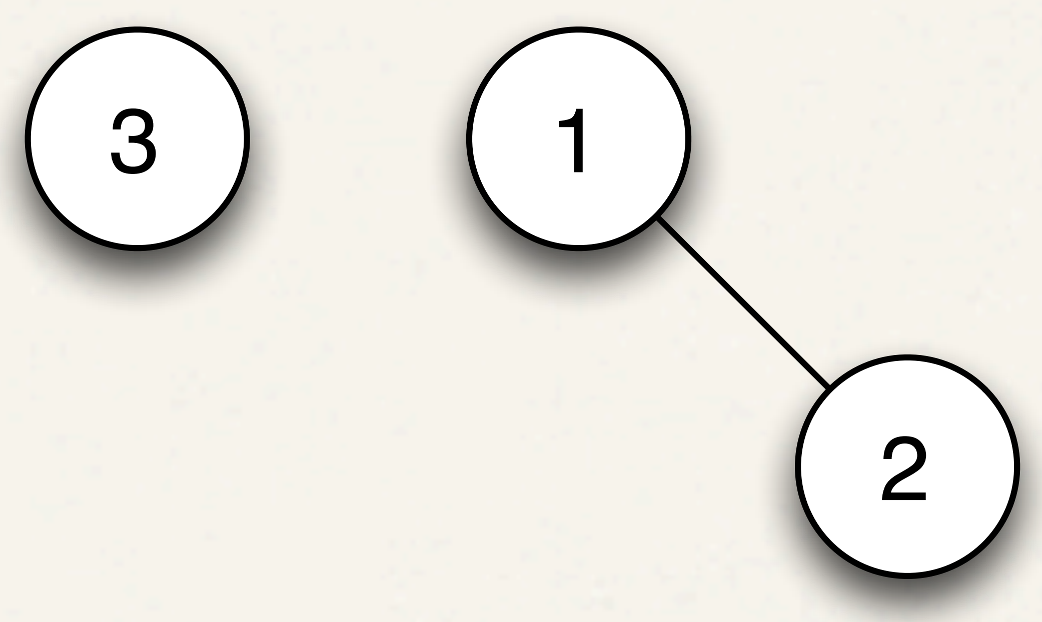
- To `insert` a node X into a binomial queue *H*:
    - Observe that a single node is a binomial tree of height 0
    - So treat X as a binomial queue
    - Merge X and *H*
- Merge operation takes $\log(N)$ time
    - Therefore so does insert

insert(1)

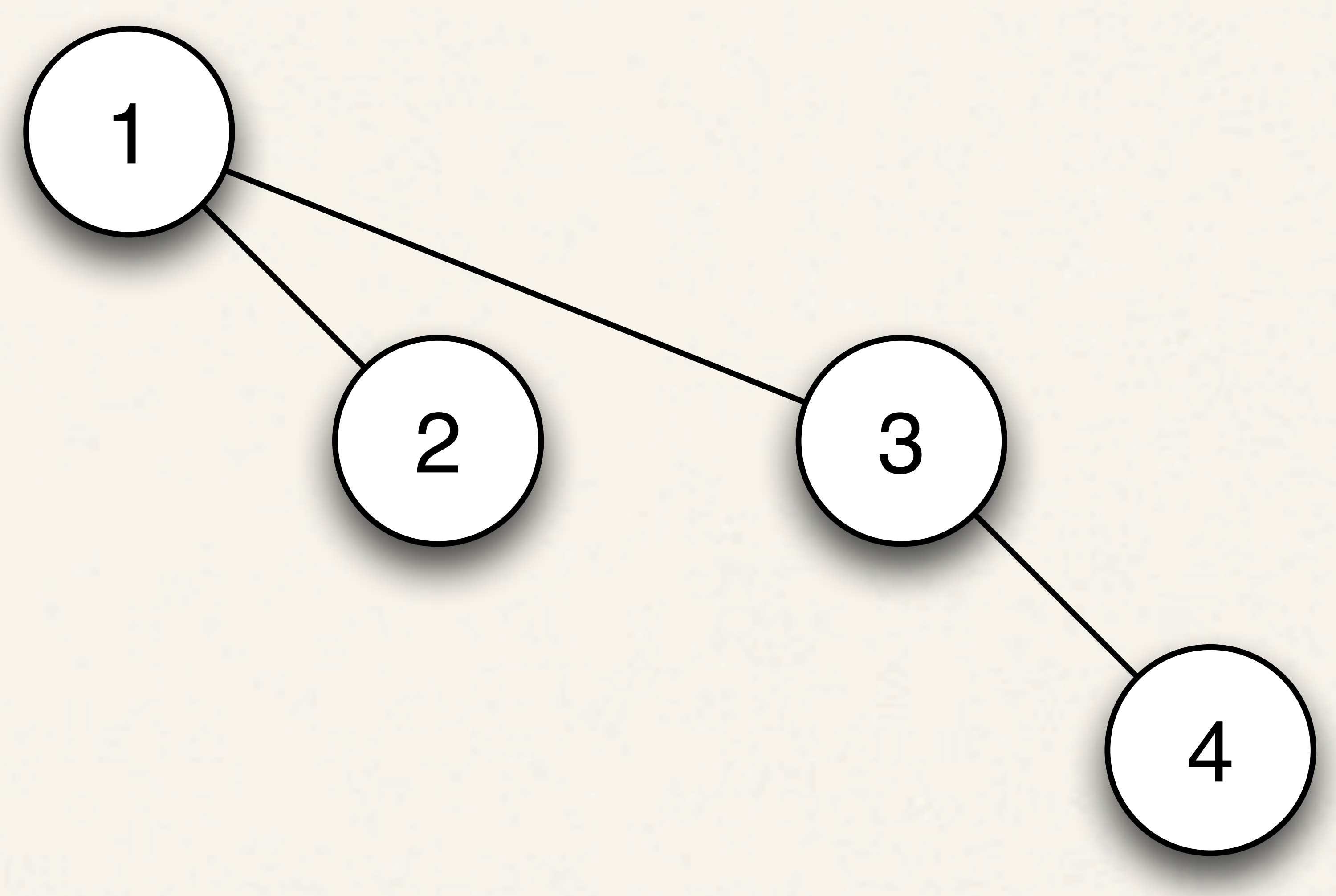
insert(2)



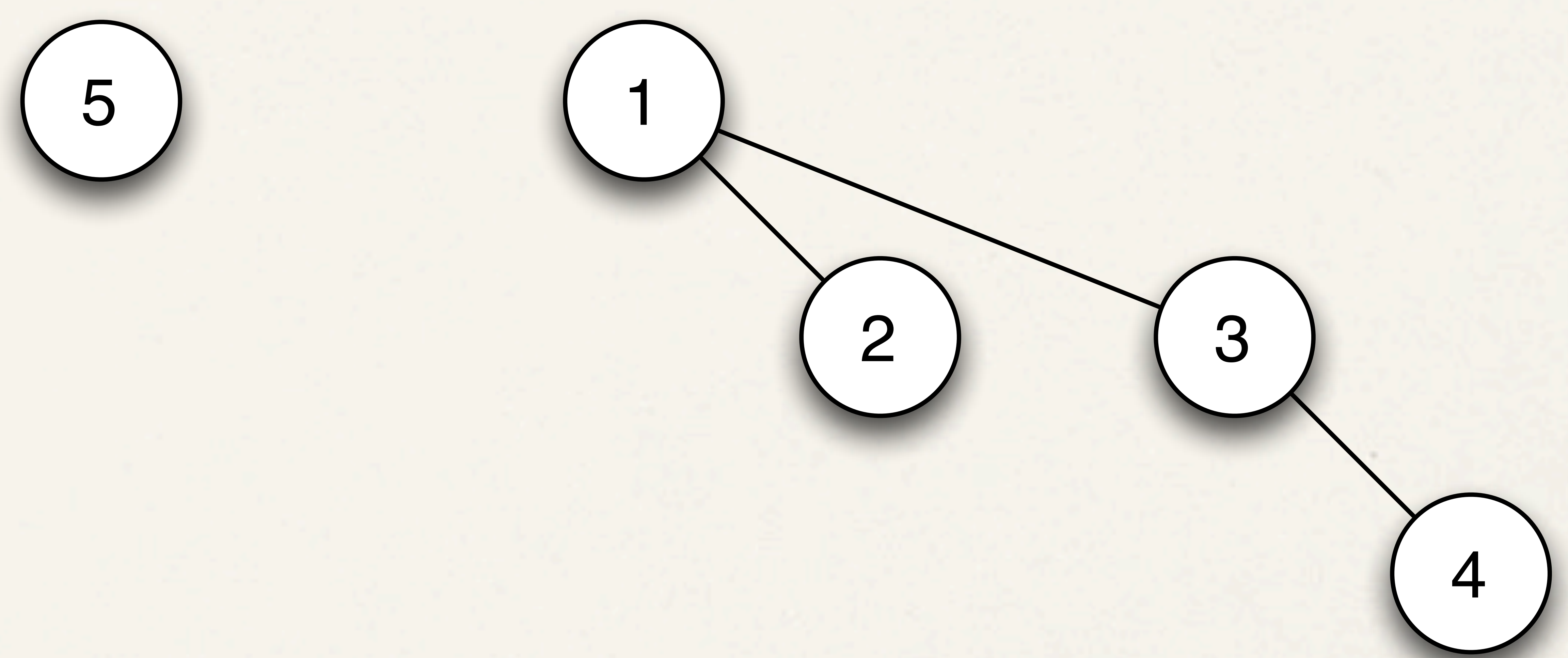
insert(3)



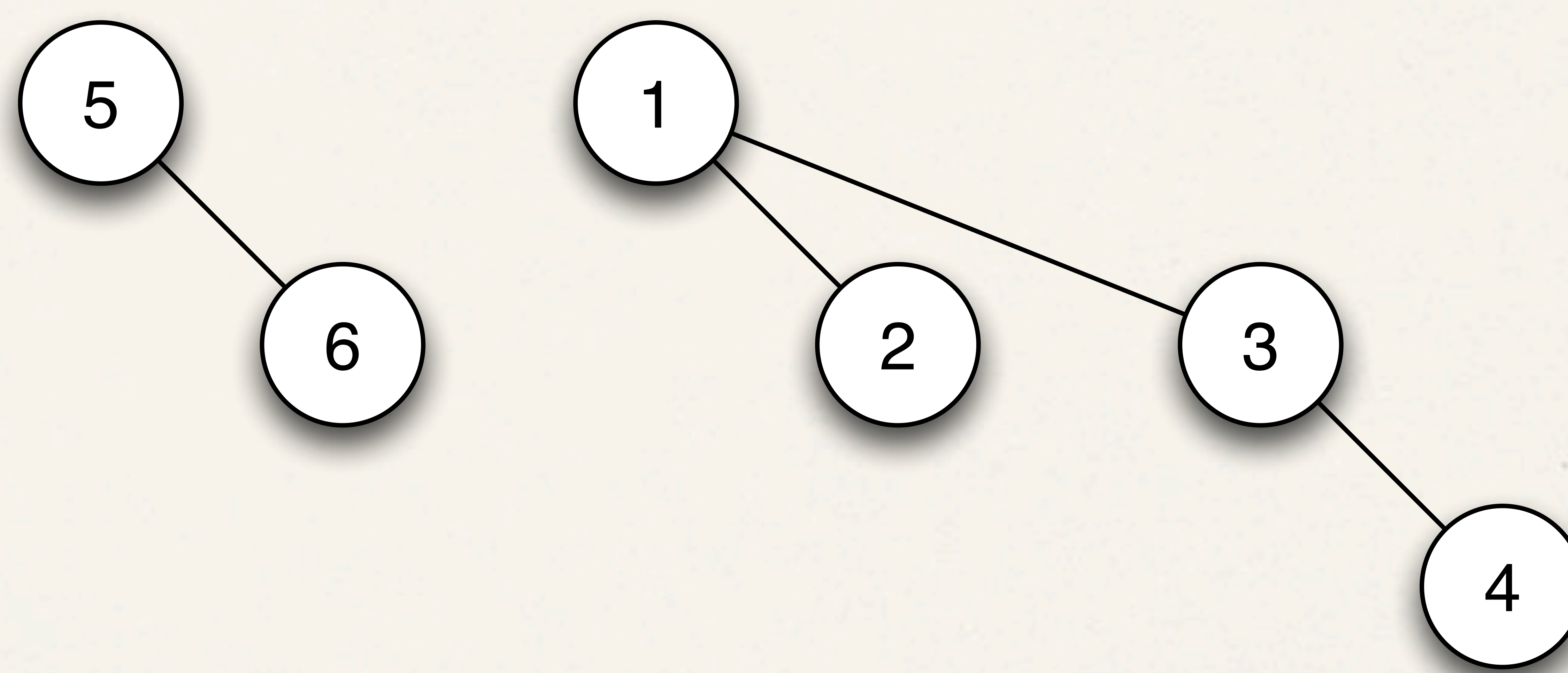
insert(4)



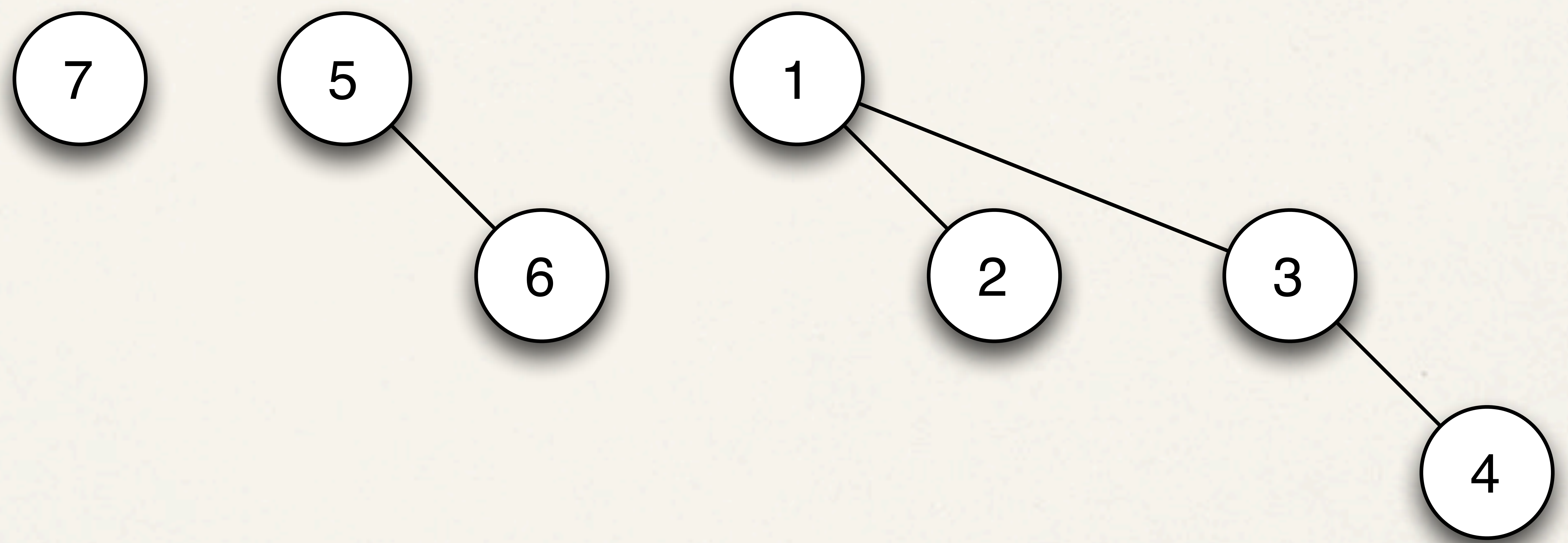
insert(5)



insert(6)



insert(7)

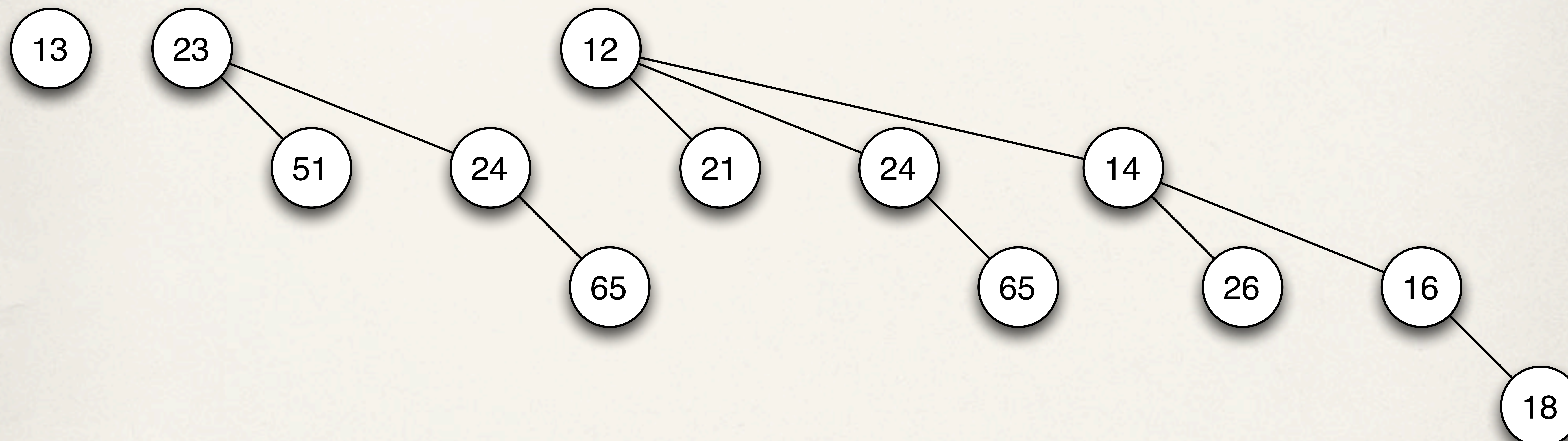


`deleteMin`

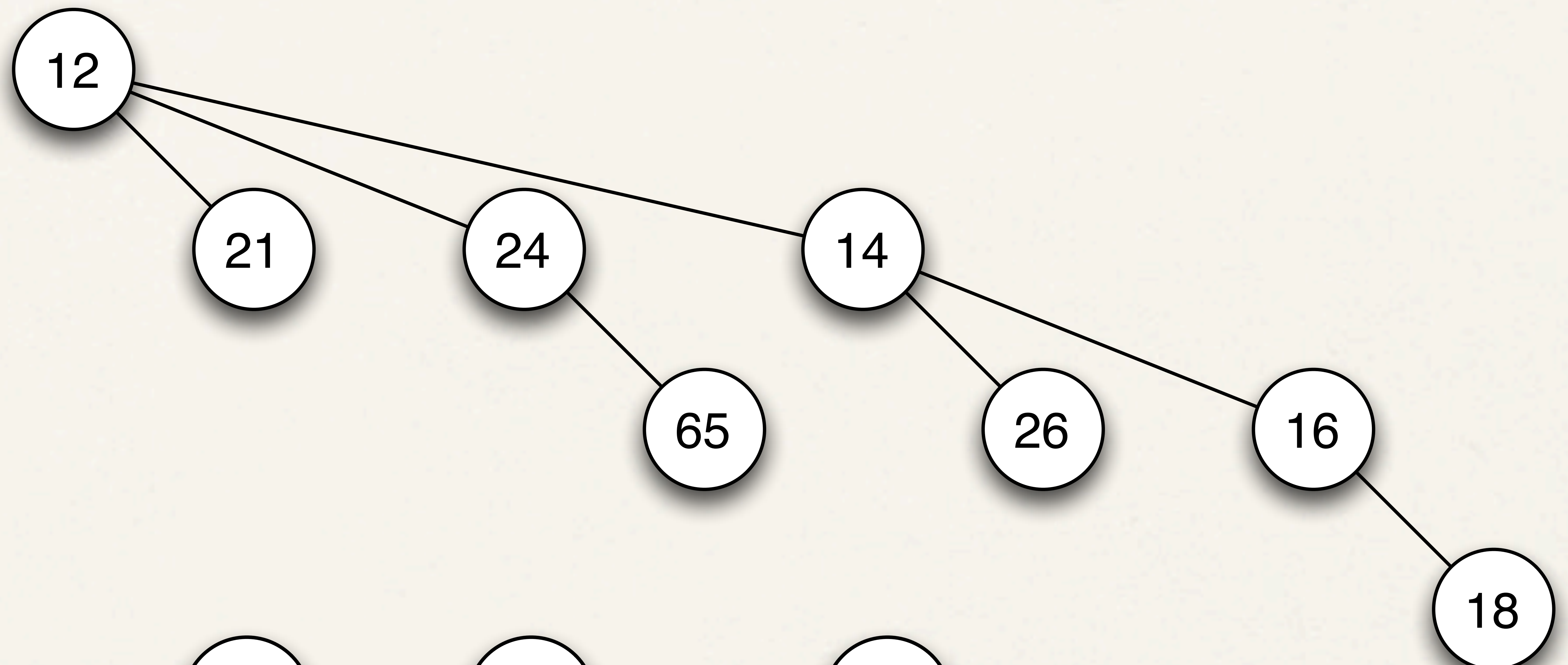
- To `deleteMin` from a binomial queue H
- Find the binomial tree with the smallest root, let this be B_k
- Remove B_k from H, leaving the rest of the trees to form queue H'
  - Delete (and return to user) the root of B_k
    - this leaves us with the children of B_k's root, which are binomial trees of size B_0, B_1, ..., B_{k-1}
    - then let the trees B_0, B_1, ..., B_{k-1} form a new binomial queue H''
- Merge H' and H'' to repair the tree

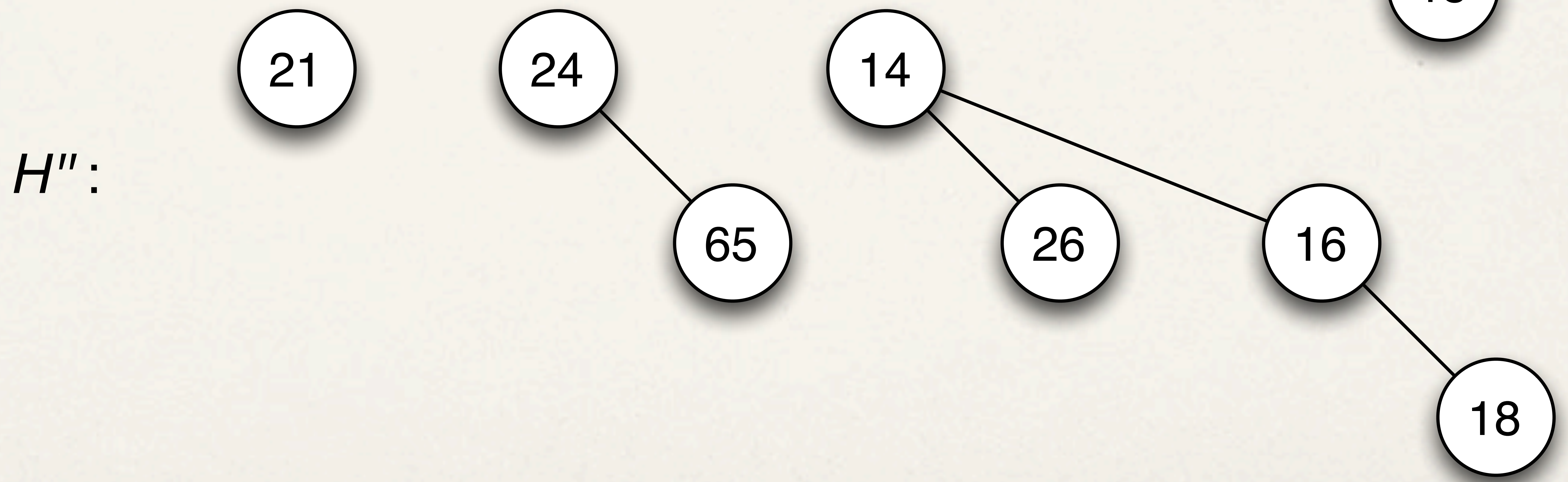
Also $O(\log N)$!

deleteMin

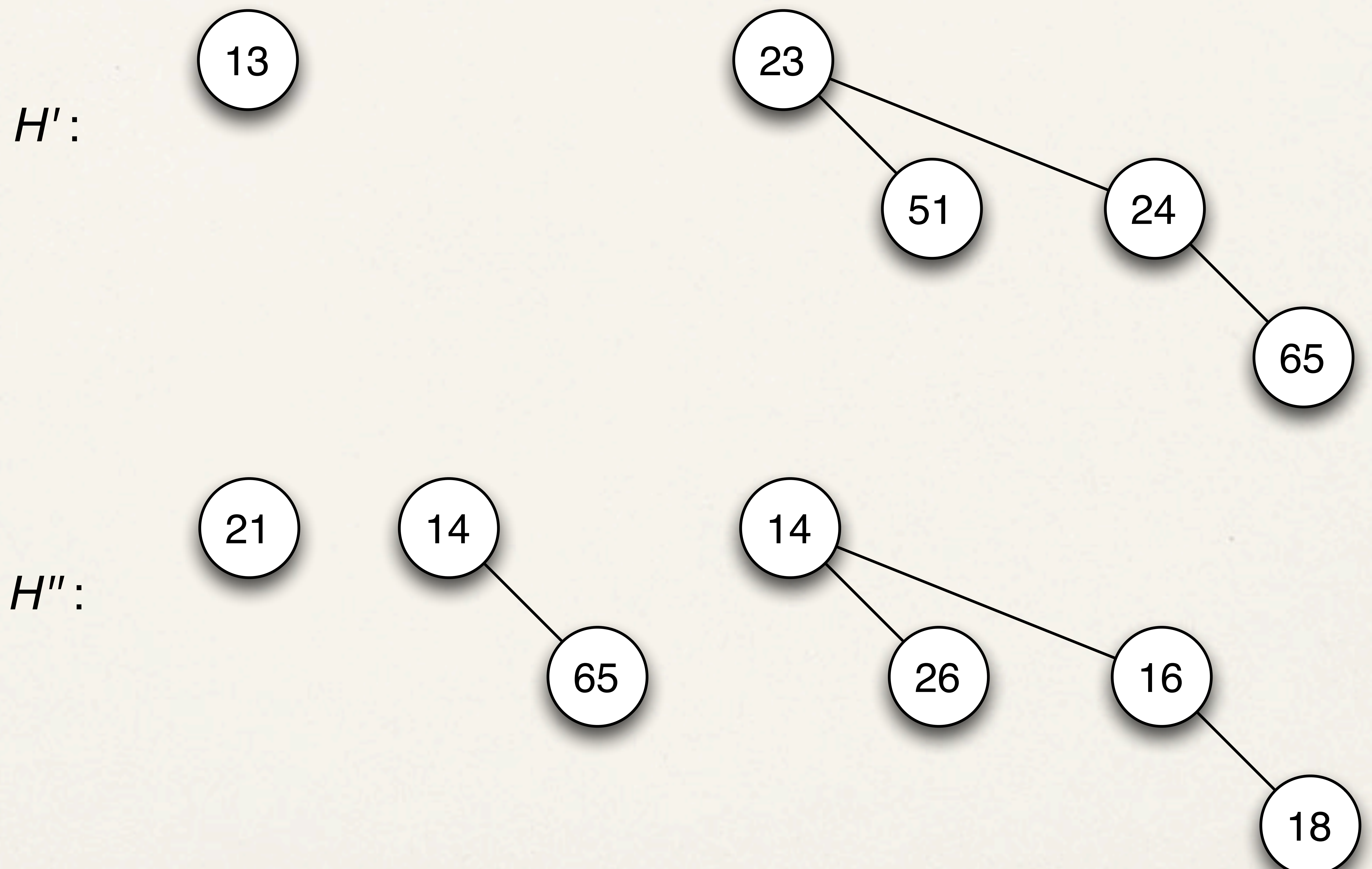


deleteMin

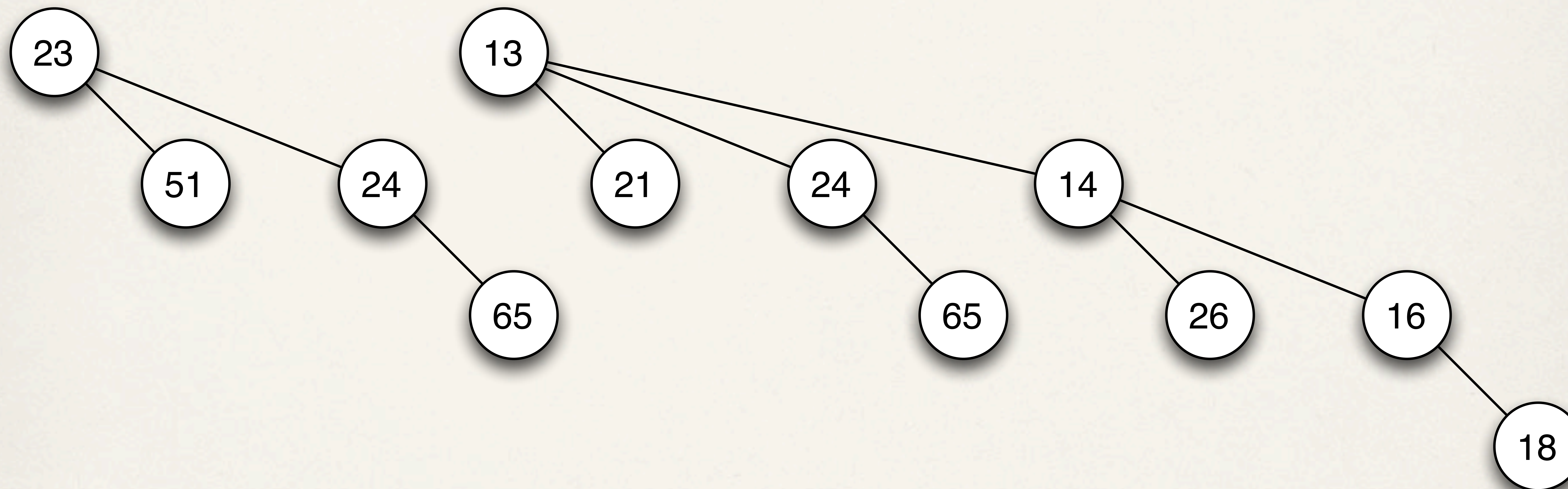




deleteMin



deleteMin



Non-Standard Operations

- `percolateUp`
  - identical to binary heap
- `decreaseKey`
  - `percolateUp` as far as root of binomial tree
- `delete` (an arbitrary node)
  - `decreaseKey` to $-\infty$, then `deleteMin`