Probabilistic Modeling and Expectation Maximization

CMSC 678

UMBC

Course Overview (so far)

Basics of Probability

- Requirements to be a distribution ("proportional to", $\propto$)
- Definitions of conditional probability, joint probability, and independence
- Bayes rule, (probability) chain rule
- Expectation (of a random variable & function)

Empirical Risk Minimization

- Gradient Descent
- Loss Functions: what is it, what does it measure, and what are some computational difficulties with them?
- Regularization: what is it, how does it work, and why might you want it?

Tasks (High Level)

- Data set splits: training vs. dev vs. test
- Classification: Posterior decoding/MAP classifier
- Classification evaluations: accuracy, precision, recall, and F scores
- Regression (vs. classification)
- Comparing supervised vs. Unsupervised Learning and their tradeoffs: why might you want to use one vs. the other, and what are some potential issues?
- Clustering: high-level goal/task, K-means as an example
- Tradeoffs among clustering evaluations

Linear Models

- Basic form of a linear model (classification or regression)
- Perceptron (simple vs. other variants, like averaged or voted)
- When you should use perceptron (what are its assumptions?)
- Perceptron as SGD

Maximum Entropy Models

- Meanings of feature functions and weights
- How to learn the weights: gradient descent
- Meaning of the maxent gradient

Neural Networks

- Relation to linear models and maxent
- Types (feedforward, CNN, RNN)
- Learning representations (e.g., "feature maps")
- What is a convolution (e.g., 1D vs 2D, high-level notions of why you might want to change padding or the width)
- How to learn: gradient descent, backprop
- Common activation functions
- Neural network regularization

Dimensionality Reduction

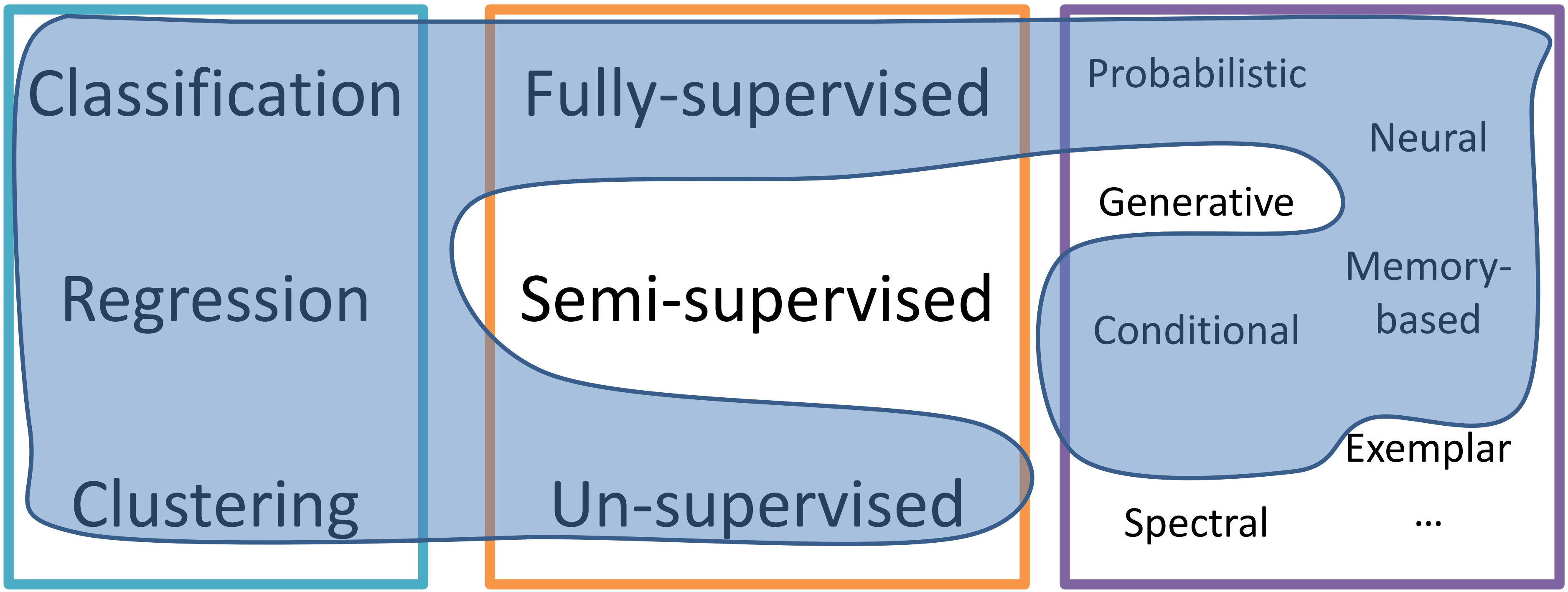
- What is the basic task & goal in dimensionality reduction?
- Dimensionality reduction tradeoffs: why might you want to, and what are some potential issues?
- Linear Discriminant Analysis vs. Principal Component Analysis: what are they trying to do, how are they similar, how do they differ?

Kernel Methods & SVMs

- Feature expansion and kernels
- Two views: maximizing a separating hyperplane margin vs. loss optimization (norm minimization)
- Non-separability & slack
- Sub-gradients

Remember from the first day: A Terminology Buffet

what we've currently sampled…



the ***task****: what* kind *of problem are you solving?*

the ***data****: amount of human input/number of labeled examples*

the ***approach****: how any data are being used*

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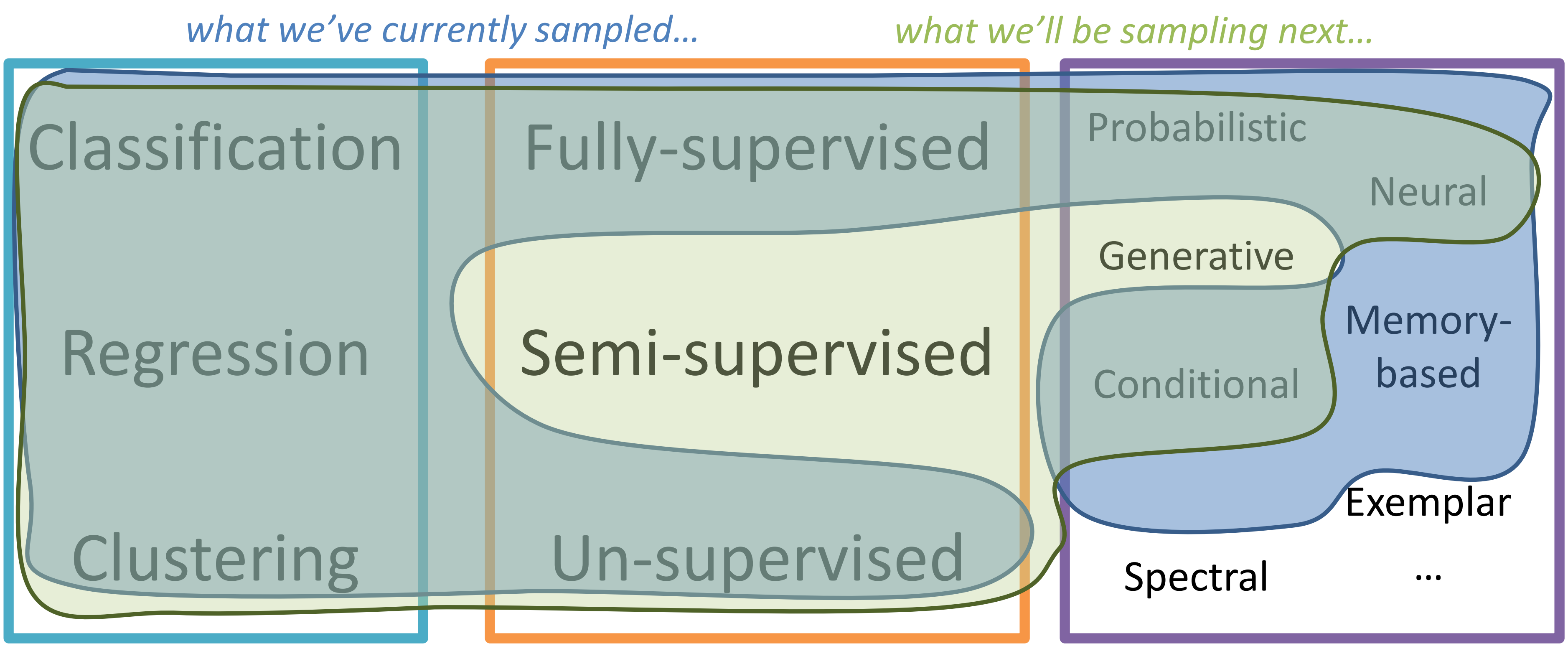

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Outline

Latent and probabilistic modeling

- Generative Modeling
- Example 1: A Model of Rolling a Die
- Example 2: A Model of Conditional Die Rolls

EM (Expectation Maximization)

- Basic idea
- Three coins example
- Why EM works

What is (Generative) Probabilistic Modeling?

So far, we've (mostly)

had *labeled* data pairs (x, y), and

built classifiers p(y | x)

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p(x, y)

Q: Where have we used p(x,y)?

A: Linear Discriminant Analysis

Or what if we only have data but no labels?

p(x)

- Like A3, Q1
- Piazza Q68

Generative Stories

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Generative stories are most often used with joint models p(x, y)…. but despite their name, generative stories are applicable to both generative and conditional models

$p(x, y)$ vs. $p(y \mid x)$: Models of our Data

$p(x, y)$ is the **joint** distribution

Two main options for estimating:

1. Directly
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Using Bayes rule *transparently* provides a **generative story** for how our data x and labels y are generated

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Two main options for estimating:

1. Directly: used when you *only* care about making the right prediction

   Examples: perceptron, logistic regression, neural networks (we've covered)

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2. Estimate the joint

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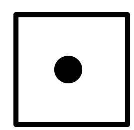
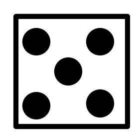
Example: Rolling a Die

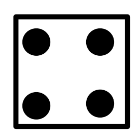
$$p(w_1, w_2, \dots, w_N) = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

Example: Rolling a Die

$$p(\overbrace{w_1, w_2, \ldots, w_N}^{\text{N different (independent) rolls}}) = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

$w_1 = 1$ 

$w_2 = 5$ 

$w_3 = 4$ 

$\ldots$

Generative Story for Rolling a Die

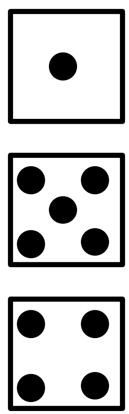
$$p\overbrace{(w_1, w_2, \ldots, w_N)}^{\text{N different (independent) rolls}} = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

$w_1 = 1$

$w_2 = 5$

$w_3 = 4$

$\ldots$



Generative Story

for roll $i = 1$ to N:

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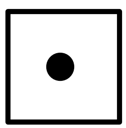
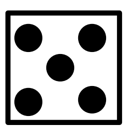
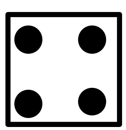
$w_1 = 1$ 

$w_2 = 5$ 

$w_3 = 4$ 

…

Generative Story

for roll $i = 1$ to N:

  $w_i \sim \text{Cat}(\theta)$

Generative Story for Rolling a Die

$$\underbrace{p(w_1, w_2, \ldots, w_N)}_{\text{N different (independent) rolls}} = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

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$\ldots$

"for each" loop becomes a product

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Calculate $p(w_i)$ according to provided distribution

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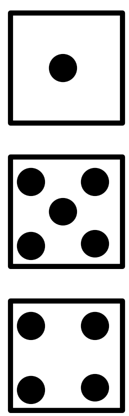

"for each" loop becomes a product

Generative Story

for roll $i = 1$ to N:

$w_i \sim \mathrm{Cat}(\theta)$

a probability distribution over 6 sides of the die

Calculate $p(w_i)$ according to provided distribution

$$\sum_{k=1}^{6} \theta_k = 1 \qquad 0 \le \theta_k \le 1, \forall k$$

Learning Parameters for the Die Model

$$p(w_1, w_2, \dots, w_N) = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

maximize (log-) likelihood to learn the probability parameters

Q: Why is maximizing log-likelihood a reasonable thing to do?

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Q: (for discrete observations) What loss function do we minimize to maximize log-likelihood?

A: Cross-entropy

Learning Parameters for the Die Model: Maximum Likelihood (Intuition)

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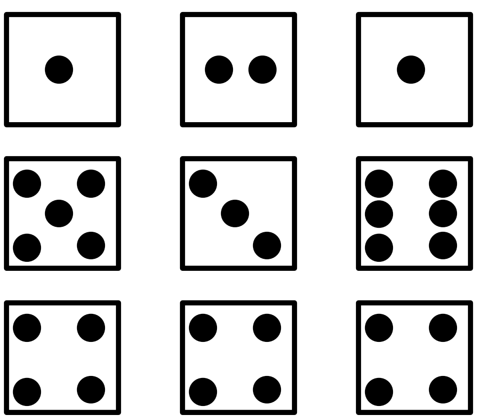
If you observe these 9 rolls…



…what are "reasonable" estimates for p(w)?

p(1) = ?

p(2) = ?

p(3) = ?

p(4) = ?

p(5) = ?

p(6) = ?

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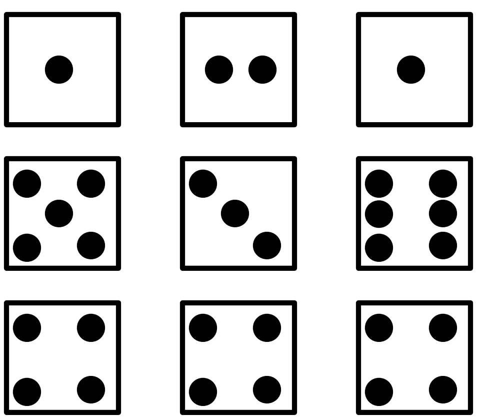

…what are “reasonable” estimates for p(w)?

p(1) = 2/9 p(2) = 1/9

p(3) = 1/9 p(4) = 3/9

p(5) = 1/9 p(6) = 1/9

maximum likelihood estimates

Learning Parameters for the Die Model: Maximum Likelihood (Math)

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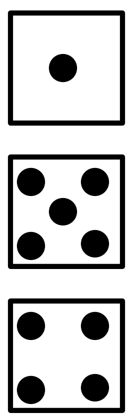
$$w_1 = 1$$

$$w_2 = 5$$

$$w_3 = 4$$

$$\ldots$$



Generative Story

for roll $i = 1$ to N:

$w_i \sim \mathrm{Cat}(\theta)$

Maximize Log-likelihood

$$\mathcal{L}(\theta) = \sum_i \log p_\theta(w_i)$$

$$= \sum_i \log \theta_{w_i}$$

Learning Parameters for the Die Model: Maximum Likelihood (Math)

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A: Just keep increasing θ_k (*we* know θ must be a distribution, but it's not specified)

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Maximize Log-likelihood (with distribution constraints)

$$\mathcal{L}(\theta) = \sum_i \log \theta_{w_i} \ \text{s.t.} \sum_{k=1}^{6} \theta_k = 1$$

(we can include the inequality constraints $0 \le \theta_k$, but it complicates the problem and, *right now*, is not needed)

solve using Lagrange multipliers

Learning Parameters for the Die Model: Maximum Likelihood (Math)

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Maximize Log-likelihood (with distribution constraints)

$$\mathcal{F}(\theta) = \sum_i \log \theta_{w_i} - \lambda\left(\sum_{k=1}^{6} \theta_k - 1\right)$$

(we can include the inequality constraints $0 \leq \theta_k$, but it complicates the problem and, *right now*, is not needed)

$$\frac{\partial \mathcal{F}(\theta)}{\partial \theta_k} = \sum_{i:w_i=k} \frac{1}{\theta_{w_i}} - \lambda \qquad \frac{\partial \mathcal{F}(\theta)}{\partial \lambda} = -\sum_{k=1}^{6} \theta_k + 1$$

Learning Parameters for the Die Model: Maximum Likelihood (Math)

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$$\theta_k = \frac{\sum_{i:w_i=k} 1}{\lambda}$$

$$\text{optimal } \lambda \text{ when } \sum_{k=1}^{6} \theta_k = 1$$

Learning Parameters for the Die Model: Maximum Likelihood (Math)

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$$\theta_k = \frac{\sum_{i:w_i=k} 1}{\sum_k \sum_{i:w_i=k} 1} = \frac{N_k}{N}$$

$$\text{optimal } \lambda \text{ when } \sum_{k=1}^{6} \theta_k = 1$$

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Example: Conditionally Rolling a Die

$$p(w_1, w_2, \ldots, w_N) = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

add complexity to better explain what we see

$$p(z_1, w_1, z_2, w_2, \ldots, z_N, w_N) = p(z_1)p(w_1|z_1)\cdots p(z_N)p(w_N|z_N)$$
$$= \prod_i p(w_i|z_i)\, p(z_i)$$

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First flip a coin…

$z_1 = T$

$z_2 = H$

…

Example: Conditionally Rolling a Die

$$p(w_1, w_2, \ldots, w_N) = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

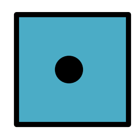
*add **complexity** to better explain what we see*

$$p(z_1, w_1, z_2, w_2, \ldots, z_N, w_N) = p(z_1)p(w_1|z_1)\cdots p(z_N)p(w_N|z_N)$$
$$= \prod_i p(w_i|z_i)\, p(z_i)$$

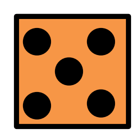
First flip a coin…

…then roll a different die depending on the coin flip

$z_1 = T$ $\quad w_1 = 1$



$z_2 = H$ $\quad w_2 = 5$



…

Learning in Conditional Die Roll Model: Maximize (Log-)Likelihood

$$p(w_1, w_2, \dots, w_N) = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

$\Big\downarrow$ *add complexity to better explain what we see*

$$p(z_1, w_1, z_2, w_2, \dots, z_N, w_N) = p(z_1)p(w_1|z_1)\cdots p(z_N)p(w_N|z_N)$$
$$= \prod_i p(w_i|z_i)\,p(z_i)$$

If you observe the z_i values, this is easy!

Learning in Conditional Die Roll Model: Maximize (Log-)Likelihood

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First: Write the Generative Story

λ = distribution over coin (z)

$\gamma^{(H)}$ = distribution for die when coin comes up heads

$\gamma^{(T)}$ = distribution for die when coin comes up tails

for item $i = 1$ to N:

  $z_i \sim \text{Bernoulli}(\lambda)$

  $w_i \sim \text{Cat}\left(\gamma^{(z_i)}\right)$

Learning in Conditional Die Roll Model: Maximize (Log-)Likelihood

$$p(z_1, w_1, z_2, w_2, \dots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

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First: Write the Generative Story

$\lambda =$ distribution over coin (z)

$\gamma^{(H)} =$ distribution for H die

$\gamma^{(T)} =$ distribution for T die

for item $i = 1$ to N:

$\quad z_i \sim \text{Bernoulli}(\lambda)$

$\quad w_i \sim \text{Cat}\left(\gamma^{(z_i)}\right)$

Second: Generative Story → Objective

$$\mathcal{F}(\theta) = \sum_i^n (\log \lambda_{z_i} + \log \gamma_{w_i}^{(z_i)})$$

$-$ Lagrange multiplier constraints

Learning in Conditional Die Roll Model: Maximize (Log-)Likelihood

$$p(z_1, w_1, z_2, w_2, \ldots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

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First: Write the Generative Story

$\lambda =$ distribution over coin (z)

$\gamma^{(H)} =$ distribution for H die

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for item $i = 1$ to N:

$\quad z_i \sim \text{Bernoulli}(\lambda)$

$\quad w_i \sim \text{Cat}\left(\gamma^{(z_i)}\right)$

Second: Generative Story → Objective

$$\mathcal{F}(\theta) = \sum_i^n \left(\log \lambda_{z_i} + \log \gamma_{w_i}^{(z_i)}\right)$$

$$-\eta\left(\sum_{k=1}^{2} \lambda_k - 1\right) - \sum_{k=1}^{2} \delta_k \left(\sum_{j=1}^{6} \gamma_j^{(k)} - 1\right)$$

Learning in Conditional Die Roll Model: Maximize (Log-)Likelihood

$$p(z_1, w_1, z_2, w_2, \ldots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

If you observe the z_i values, this is easy!

But if you don't observe the z_i values, this is not easy!

First: Write the Generative Story

λ = distribution over coin (z)

$\gamma^{(H)}$ = distribution for H die

$\gamma^{(T)}$ = distribution for T die

for item $i = 1$ to N:

  $z_i \sim \text{Bernoulli}(\lambda)$

  $w_i \sim \text{Cat}\left(\gamma^{(z_i)}\right)$

Second: Generative Story → Objective

$$\mathcal{F}(\theta) = \sum_i^n \left(\log \lambda_{z_i} + \log \gamma_{w_i}^{(z_i)}\right)$$

$$-\eta\left(\sum_{k=1}^{2} \lambda_k - 1\right) - \sum_{k=1}^{2} \delta_k \left(\sum_{j=1}^{6} \gamma_j^{(k)} - 1\right)$$

Example: Conditionally Rolling a Die

$$p(z_1, w_1, z_2, w_2, \ldots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

goal: maximize (log-)likelihood

we don't actually observe these z values
we just see the items w

if we *did* observe *z*, estimating the probability parameters would be easy…
but we don't! :(

Example: Conditionally Rolling a Die

$$p(z_1, w_1, z_2, w_2, \dots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

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Example: Conditionally Rolling a Die

$$p(z_1, w_1, z_2, w_2, \ldots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

we don't actually observe these z values

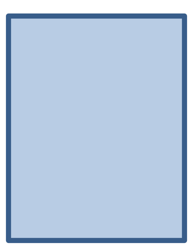
goal: maximize ***marginalized*** (log-)likelihood

Example: Conditionally Rolling a Die

$$p(z_1, w_1, z_2, w_2, \dots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

we don't actually observe these z values

goal: maximize ***marginalized*** (log-)likelihood



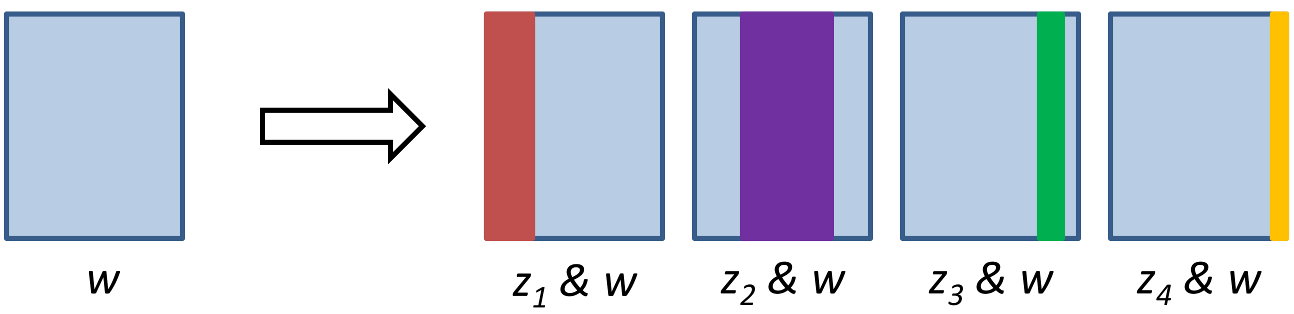
w

Example: Conditionally Rolling a Die

$$p(z_1, w_1, z_2, w_2, \ldots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

we don't actually observe these z values

goal: maximize ***marginalized*** (log-)likelihood

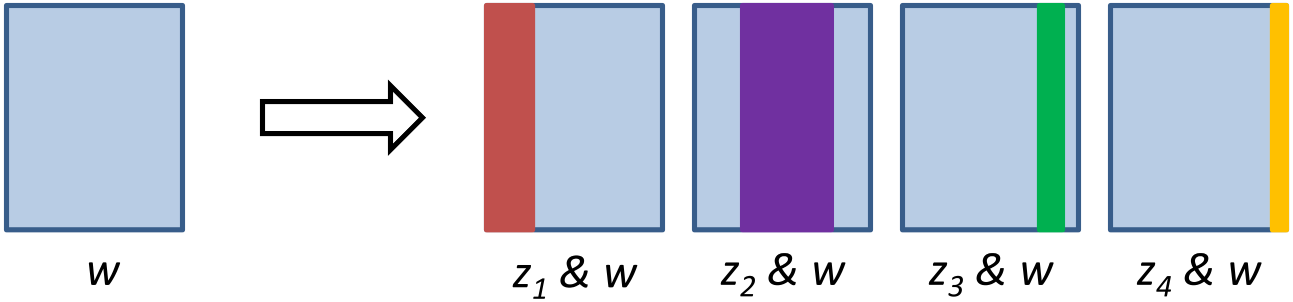


Example: Conditionally Rolling a Die

$$p(z_1, w_1, z_2, w_2, \dots, z_N, w_N) = \prod_i p(w_i | z_i)\, p(z_i)$$

we don't actually observe these z values

goal: maximize ***marginalized*** (log-)likelihood

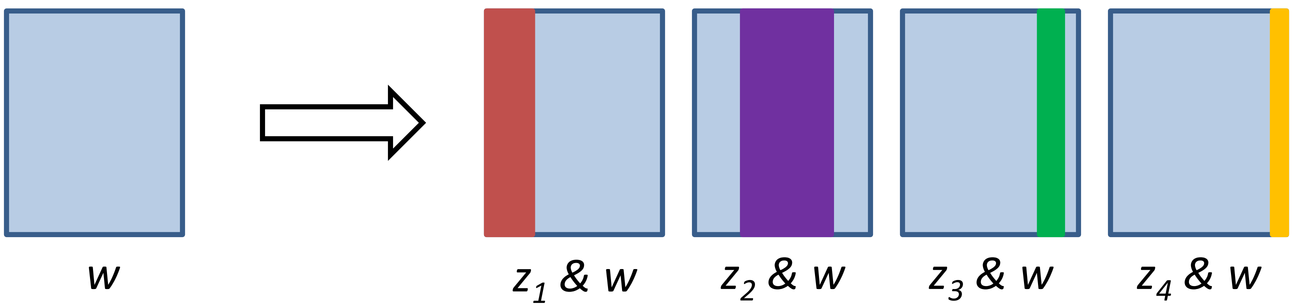


$$p(w_1, w_2, \dots, w_N) = \left(\sum_{z_1} p(z_1, w)\right)\left(\sum_{z_2} p(z_2, w)\right) \cdots \left(\sum_{z_N} p(z_N, w)\right)$$

Example: Conditionally Rolling a Die

$$p(z_1, w_1, z_2, w_2, \ldots, z_N, w_N) = p(z_1)p(w_1|z_1) \cdots p(z_N)p(w_N|z_N)$$

goal: maximize ***marginalized*** (log-)likelihood



$$p(w_1, w_2, \ldots, w_N) = \left(\sum_{z_1} p(z_1, w)\right)\left(\sum_{z_2} p(z_2, w)\right) \cdots \left(\sum_{z_N} p(z_N, w)\right)$$

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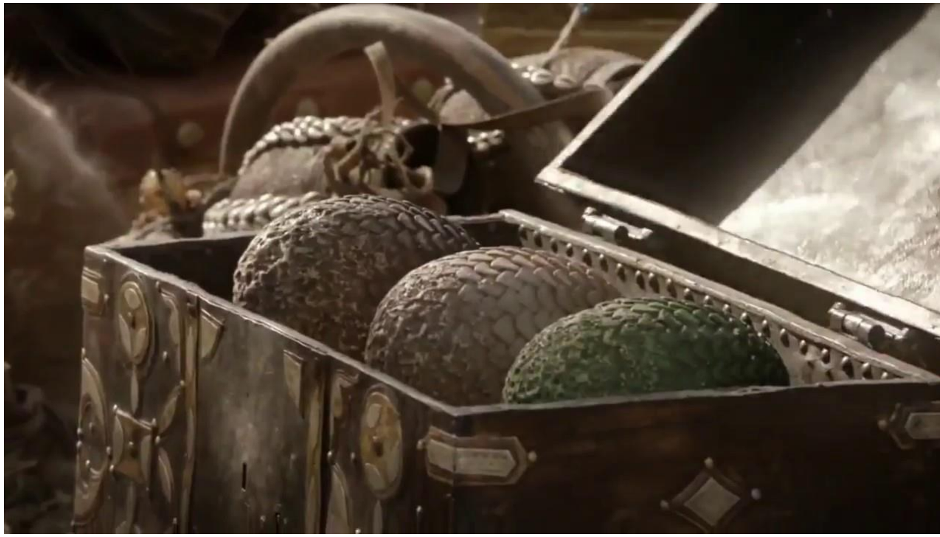

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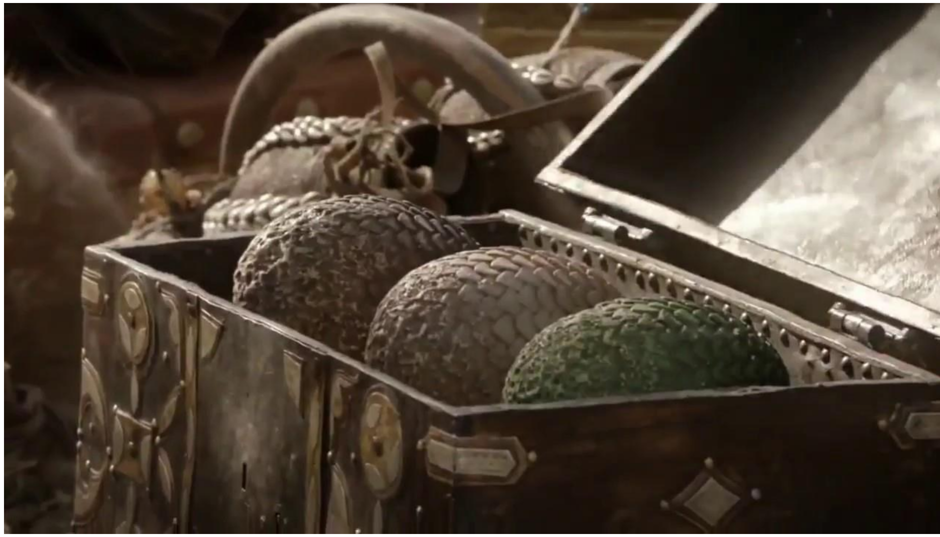

if we *did* observe z, estimating the
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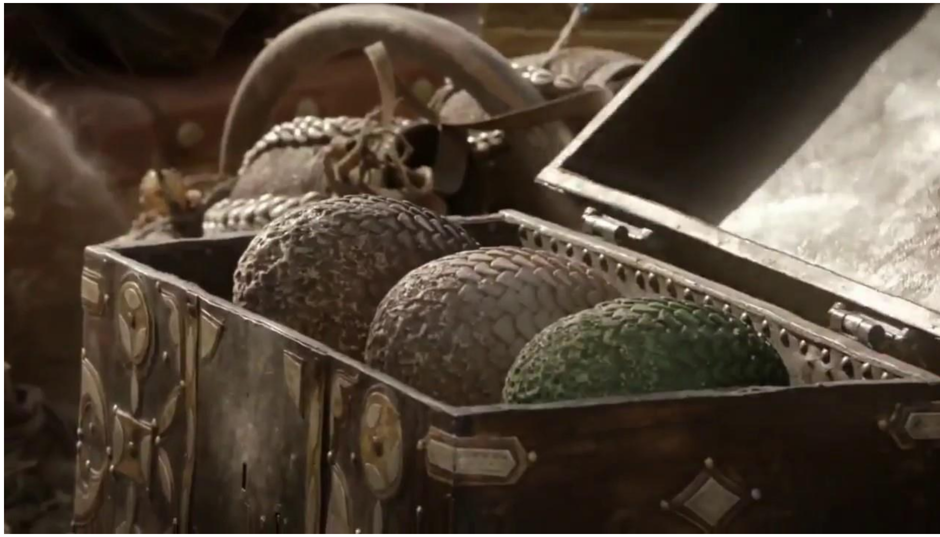


if we *did* observe z, estimating the probability parameters would be easy… but we don't! :(

Expectation Maximization: give you model estimation the needed "spark"

Outline

Latent and probabilistic modeling

- Generative Modeling
- Example 1: A Model of Rolling a Die
- Example 2: A Model of Conditional Die Rolls

EM (Expectation Maximization)

- Basic idea
- Three coins example
- Why EM works

Expectation Maximization (EM)

0. Assume *some* value for your parameters

Two step, iterative algorithm

1. E-step: count under uncertainty (compute expectations)

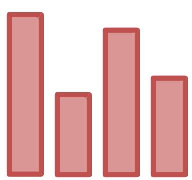
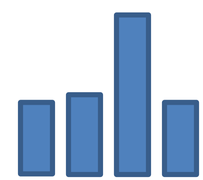
2. M-step: maximize log-likelihood, assuming these uncertain counts

Expectation Maximization (EM): E-step

0. Assume *some* value for your parameters

Two step, iterative algorithm

1. E-step: count under uncertainty, assuming these parameters

$p(z_i)$  $\Rightarrow$ $\text{count}(z_i, w_i)$ 

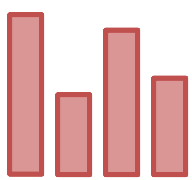
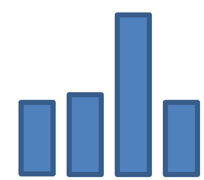
2. M-step: maximize log-likelihood, assuming these uncertain counts

Expectation Maximization (EM): E-step

0. Assume *some* value for your parameters

Two step, iterative algorithm

1. E-step: count under uncertainty, assuming these parameters

$p(z_i)$  $\Longrightarrow$ $\text{count}(z_i, w_i)$ 

2. M- … se

uncer …

We've already seen this type of counting, when computing the gradient in maxent models.

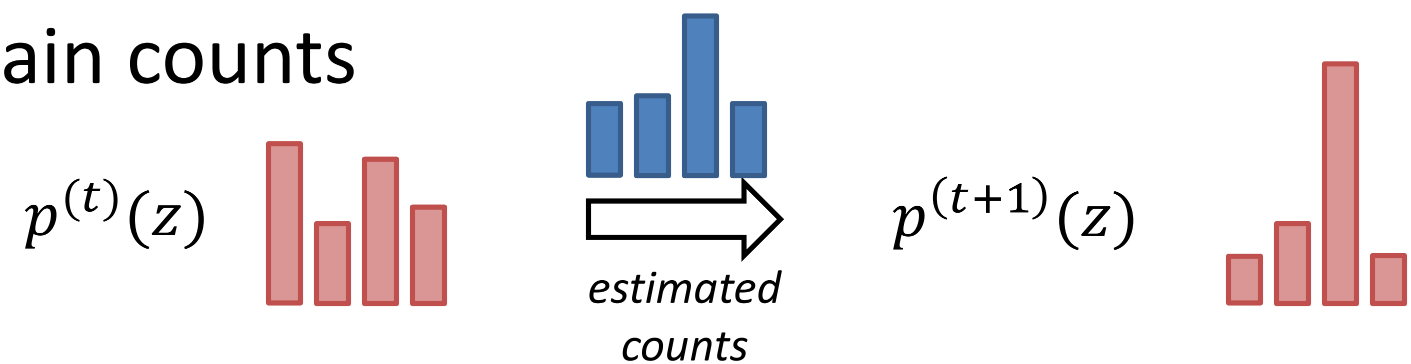
Expectation Maximization (EM): M-step

0. Assume *some* value for your parameters

Two step, iterative algorithm

1. E-step: count under uncertainty, assuming these parameters

2. M-step: maximize log-likelihood, assuming these uncertain counts



EM Math

$\max$ the average log-likelihood of our complete data (z, w), averaged across all z and according to how likely our *current* model thinks z is

EM Math

maximize the average log-likelihood of our complete data (z, w), averaged across all z and according to how likely our *current* model thinks z is

$$\max_{\theta} \mathbb{E}_{z \sim p_{\theta^{(t)}}(\cdot|w)} [\log p_{\theta}(z, w)]$$

EM Math

maximize the average log-likelihood of our complete data (z, w), averaged across all z and according to how likely our *current* model thinks z is

$$\max_{\theta} \mathbb{E}_{z \sim p_{\theta^{(t)}}(\cdot|w)} \left[\log p_{\theta}(z, w) \right]$$

EM Math

maximize the average log-likelihood of our complete data (z, w), averaged across all z and according to how likely our *current* model thinks z is

$$\max_{\theta} \mathbb{E}_{z \sim p_{\theta^{(t)}}(\cdot|w)} \left[\log p_{\theta}(z, w) \right]$$

current parameters

posterior distribution

EM Math

maximize the average log-likelihood of our complete data (z, w), averaged across all z and according to how likely our *current* model thinks z is

current parameters

$$\max_{\theta} \mathbb{E}_{z \sim p_{\theta^{(t)}}(\cdot|w)} \left[\log p_{\theta}(z, w) \right]$$

new parameters

posterior distribution

new parameters

EM Math

maximize the average log-likelihood of our complete data (z, w), averaged across all z and according to how likely our *current* model thinks z is

current parameters

$$\max_{\theta} \mathbb{E}_{z \sim p_{\theta^{(t)}}(\cdot|w)} [\log p_{\theta}(z, w)]$$

new parameters

posterior distribution

new parameters

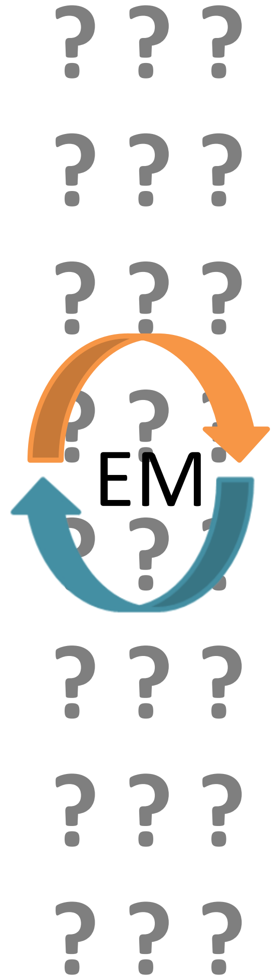
E-step: count under uncertainty

M-step: maximize log-likelihood

Why EM? Un-Supervised Learning

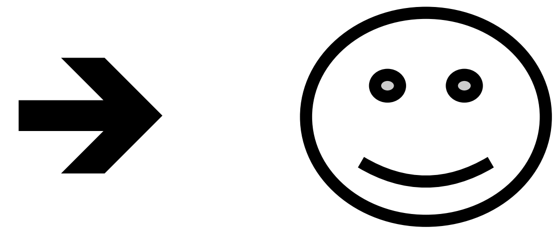
NO labeled data:

- human annotated
- relatively small/few examples



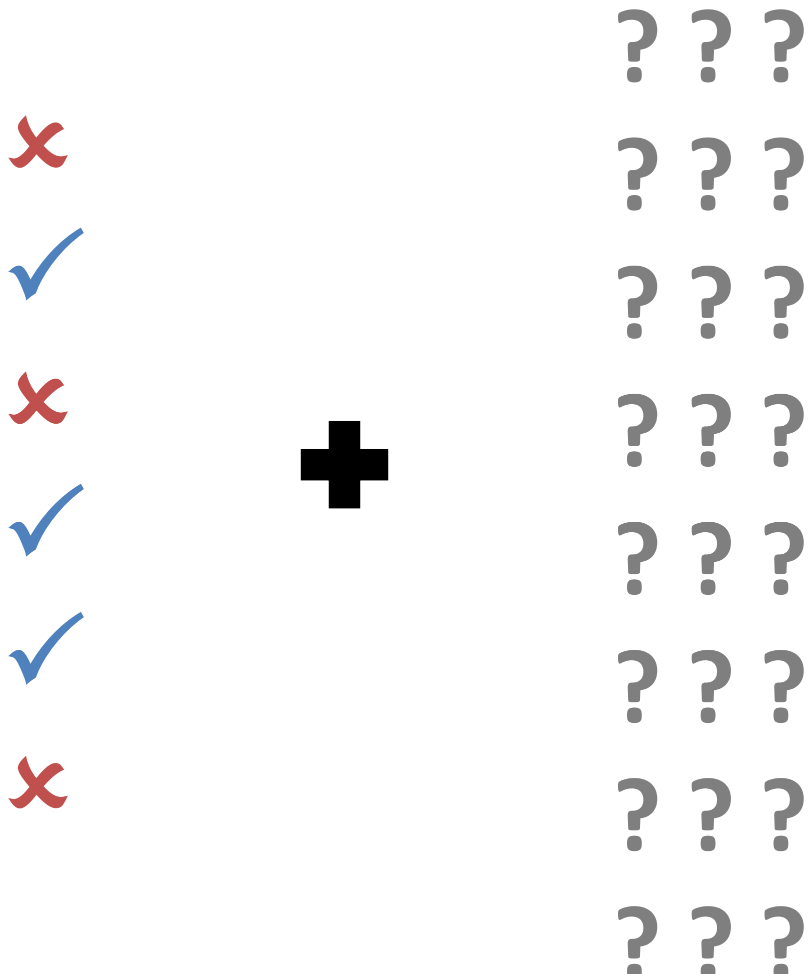
unlabeled data:

- raw; not annotated
- plentiful



EM/generative models in this case can be seen as a type of clustering

Why EM? Semi-Supervised Learning



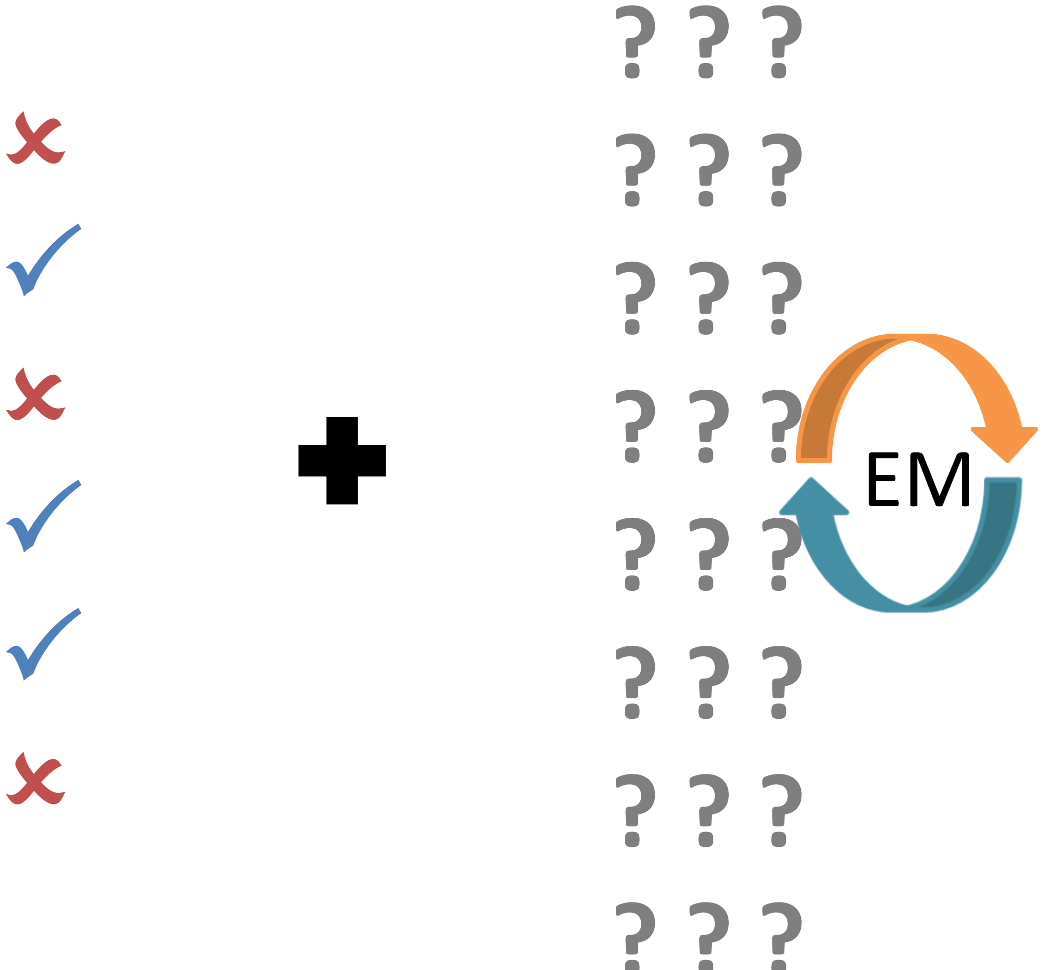
labeled data:

- human annotated
- relatively small/few examples

unlabeled data:

- raw; not annotated
- plentiful

Why EM? Semi-Supervised Learning



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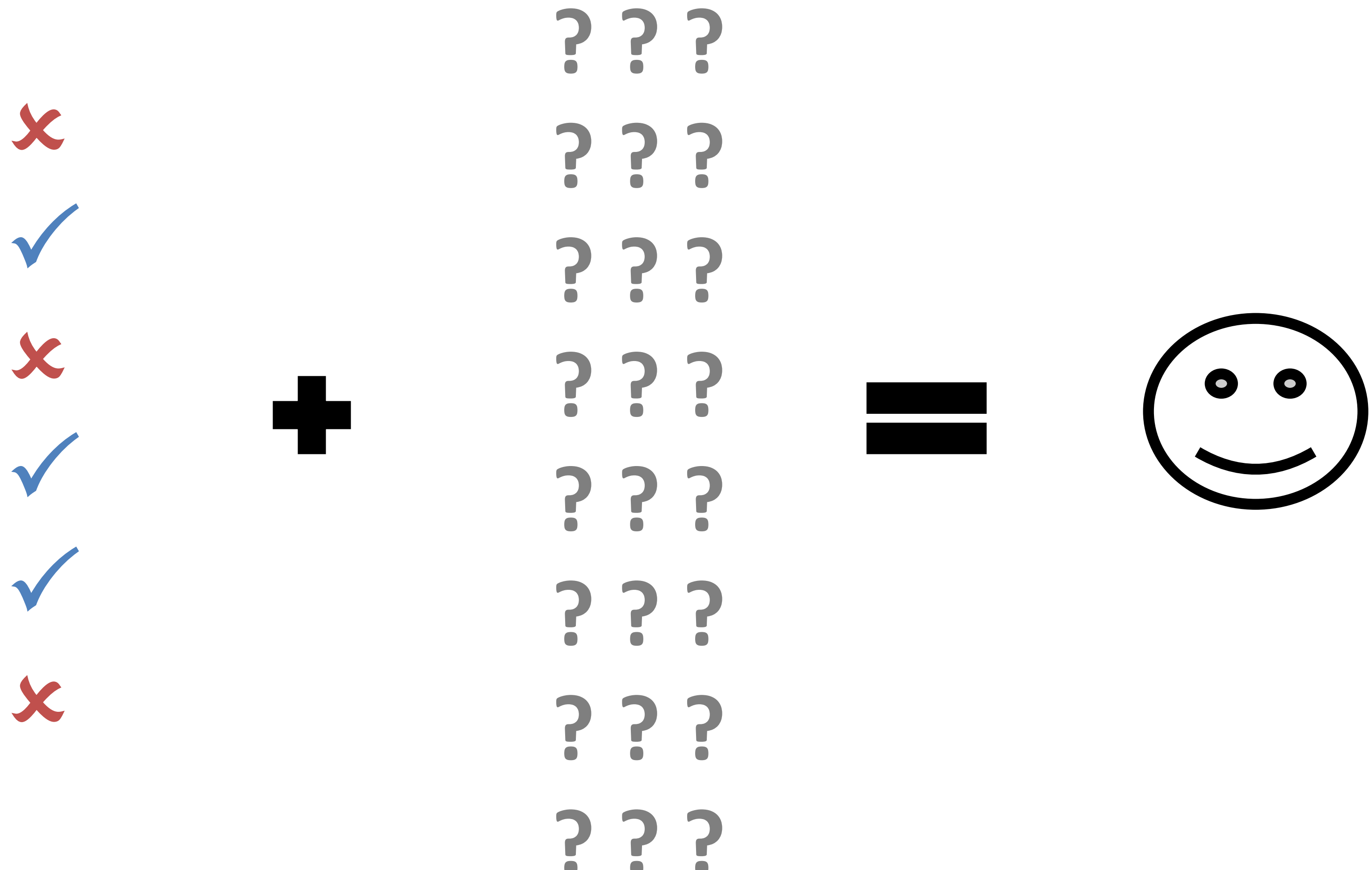
Why EM? Semi-Supervised Learning



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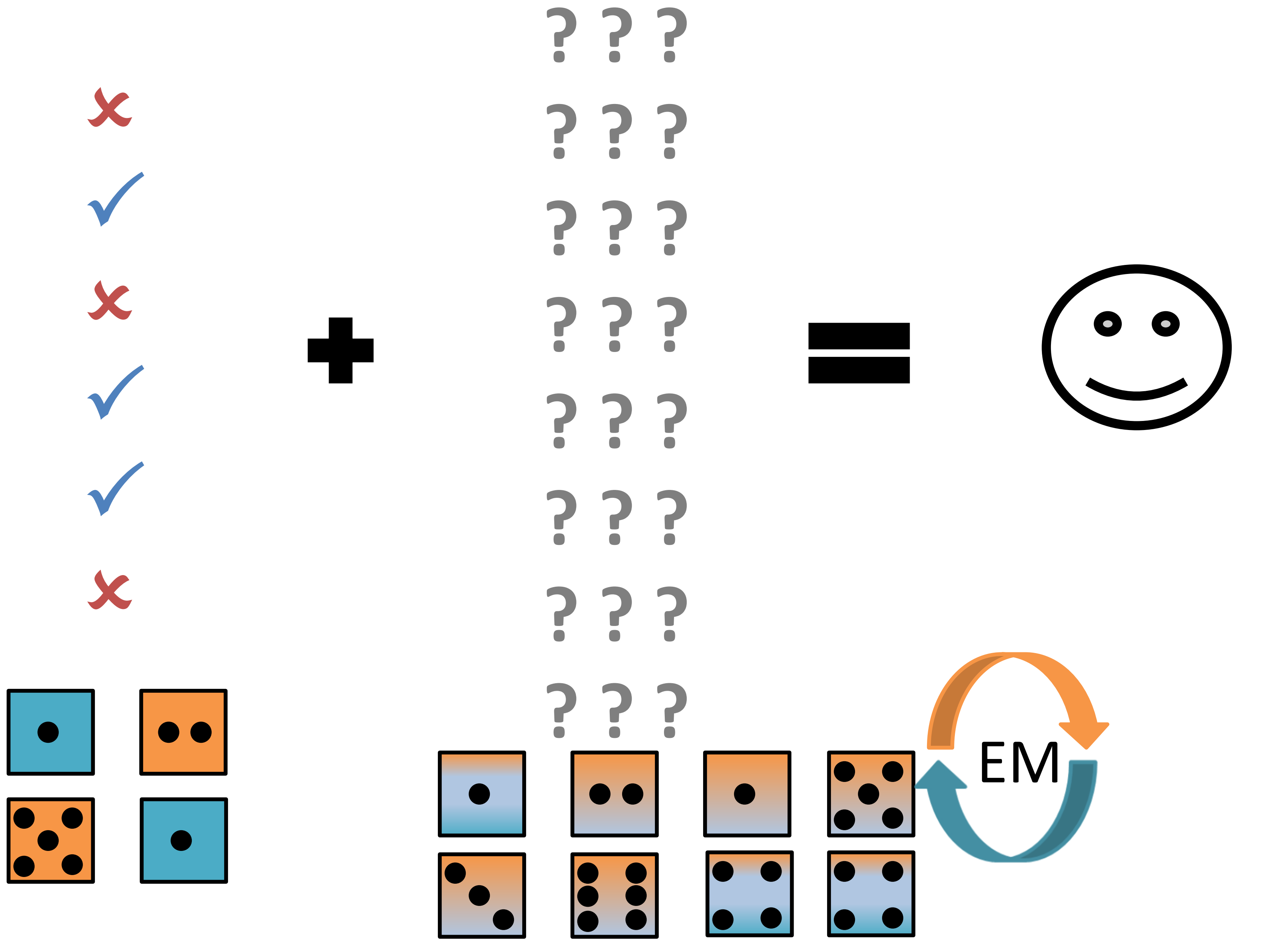
Why EM? Semi-Supervised Learning



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Three Coins Example

Imagine three coins

Flip 1st coin (penny)

If heads: flip 2nd coin (dollar coin)

If tails: flip 3rd coin (dime)

Three Coins Example

Imagine three coins

Flip 1st coin (penny)

don't observe this

If heads: flip 2nd coin (dollar coin)

only observe these (record heads vs. tails outcome)

If tails: flip 3rd coin (dime)

Three Coins Example

Imagine three coins

Flip 1st coin (penny)

unobserved:
part of speech?
genre?

If heads: flip 2nd coin (dollar coin)

If tails: flip 3rd coin (dime)

observed:
a, *b*, *e*, etc.
We run the code, vs.
The *run* failed

Three Coins Example

Imagine three coins

Flip 1st coin (penny)

$p(\text{heads}) = \lambda$ $\quad p(\text{tails}) = 1 - \lambda$

If heads: flip 2nd coin (dollar coin)

$p(\text{heads}) = \gamma$ $\quad p(\text{tails}) = 1 - \gamma$

If tails: flip 3rd coin (dime)

$p(\text{heads}) = \psi$ $\quad p(\text{tails}) = 1 - \psi$

Three Coins Example

Imagine three coins

$p(\text{heads}) = \lambda$
$p(\text{tails}) = 1 - \lambda$

$p(\text{heads}) = \gamma$
$p(\text{tails}) = 1 - \gamma$

$p(\text{heads}) = \psi$
$p(\text{tails}) = 1 - \psi$

Three parameters to estimate: λ, γ, and ψ

Generative Story for Three Coins

$$p(w_1, w_2, \dots, w_N) = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

$\Downarrow$ *add complexity to better explain what we see*

$$p(z_1, w_1, z_2, w_2, \dots, z_N, w_N) = p(z_1)p(w_1|z_1)\cdots p(z_N)p(w_N|z_N)$$

$$= \prod_i p(w_i|z_i)\, p(z_i)$$

$p(\text{heads}) = \lambda$

$p(\text{tails}) = 1 - \lambda$

$p(\text{heads}) = \gamma$

$p(\text{tails}) = 1 - \gamma$

$p(\text{heads}) = \psi$

$p(\text{tails}) = 1 - \psi$

Generative Story

λ = distribution over penny

γ = distribution for dollar coin

ψ = distribution over dime

for item $i = 1$ to N:

$z_i \sim \text{Bernoulli}(\lambda)$

if $z_i = H$: $w_i \sim \text{Bernoulli}(\gamma)$

else: $w_i \sim \text{Bernoulli}(\psi)$

Three Coins Example

H H T H T H

H T H T T T

If *all* flips were observed

$$p(\text{heads}) = \lambda \qquad p(\text{heads}) = \gamma \qquad p(\text{heads}) = \psi$$

$$p(\text{tails}) = 1 - \lambda \qquad p(\text{tails}) = 1 - \gamma \qquad p(\text{tails}) = 1 - \psi$$

Three Coins Example

H H T H T H

H T H T T T

If *all* flips were observed

$p(\text{heads}) = \lambda$ $\quad$ $p(\text{heads}) = \gamma$ $\quad$ $p(\text{heads}) = \psi$

$p(\text{tails}) = 1 - \lambda$ $\quad$ $p(\text{tails}) = 1 - \gamma$ $\quad$ $p(\text{tails}) = 1 - \psi$

$$p(\text{heads}) = \frac{4}{6} \qquad p(\text{heads}) = \frac{1}{4} \qquad p(\text{heads}) = \frac{1}{2}$$

$$p(\text{tails}) = \frac{2}{6} \qquad p(\text{tails}) = \frac{3}{4} \qquad p(\text{tails}) = \frac{1}{2}$$

Three Coins Example

~~H H T H T H~~

H T H T T T

But not all flips are observed → *set* parameter values

$p(\text{heads}) = \lambda = .6$ $p(\text{heads}) = .8$ $p(\text{heads}) = .6$

$p(\text{tails}) = .4$ $p(\text{tails}) = .2$ $p(\text{tails}) = .4$

Three Coins Example

~~H H T H T H~~

H T H T T T

But not all flips are observed → *set* parameter values

$p(\text{heads}) = \lambda = .6$ $\quad$ $p(\text{heads}) = .8$ $\quad$ $p(\text{heads}) = .6$

$p(\text{tails}) = .4$ $\quad$ $p(\text{tails}) = .2$ $\quad$ $p(\text{tails}) = .4$

Use these values to compute posteriors

$$p(\text{heads} \mid \text{observed item H}) = \frac{p(\text{heads \& H})}{p(\text{H})}$$

$$p(\text{heads} \mid \text{observed item T}) = \frac{p(\text{heads \& T})}{p(\text{T})}$$

Three Coins Example

$\cancel{H\ H\ T\ H\ T\ H}$

H T H T T T

But not all flips are observed → *set* parameter values

$p(\text{heads}) = \lambda = .6$ $\quad$ $p(\text{heads}) = .8$ $\quad$ $p(\text{heads}) = .6$

$p(\text{tails}) = .4$ $\quad$ $p(\text{tails}) = .2$ $\quad$ $p(\text{tails}) = .4$

Use these values to compute posteriors

rewrite joint using Bayes rule

$$p(\text{heads} \mid \text{observed item H}) = \frac{p(\text{H} \mid \text{heads})\,p(\text{heads})}{p(\text{H})}$$

marginal likelihood

Three Coins Example

~~*H H T H T H*~~

H T H T T T

But not all flips are observed → *set* parameter values

$p(\text{heads}) = \lambda = .6$ $\quad$ $p(\text{heads}) = .8$ $\quad$ $p(\text{heads}) = .6$

$p(\text{tails}) = .4$ $\quad$ $p(\text{tails}) = .2$ $\quad$ $p(\text{tails}) = .4$

Use these values to compute posteriors

$$p(\text{heads} \mid \text{observed item H}) = \frac{p(\text{H} \mid \text{heads})p(\text{heads})}{p(\text{H})}$$

$p(\text{H} \mid \text{heads}) = .8$ $\quad$ $p(\text{T} \mid \text{heads}) = .2$

Three Coins Example

~~H H T H T H~~

H T H T T T

But not all flips are observed → *set* parameter values

$p(\text{heads}) = \lambda = .6$ $\quad$ $p(\text{heads}) = .8$ $\quad$ $p(\text{heads}) = .6$

$p(\text{tails}) = .4$ $\quad$ $p(\text{tails}) = .2$ $\quad$ $p(\text{tails}) = .4$

Use these values to compute posteriors

$$p(\text{heads} \mid \text{observed item H}) = \frac{p(\text{H} \mid \text{heads})p(\text{heads})}{p(\text{H})}$$

$p(\text{H} \mid \text{heads}) = .8$ $\quad$ $p(\text{T} \mid \text{heads}) = .2$

$$p(\text{H}) = p(\text{H} \mid \text{heads}) * p(\text{heads}) + p(\text{H} \mid \text{tails}) * p(\text{tails})$$
$$= .8 * .6 + .6 * .4$$

Three Coins Example

~~*H H T H T H*~~

H T H T T T

Use posteriors to update parameters

$$p(\text{heads} \mid \text{obs. H}) = \frac{p(\text{H} \mid \text{heads})p(\text{heads})}{p(\text{H})} = \frac{.8 * .6}{.8 * .6 + .6 * .4} \approx 0.667$$

$$p(\text{heads} \mid \text{obs. T}) = \frac{p(\text{T} \mid \text{heads})p(\text{heads})}{p(\text{T})} = \frac{.2 * .6}{.2 * .6 + .6 * .4} \approx 0.334$$

Q: Is p(heads | obs. H) + p(heads | obs. T) = 1?

Three Coins Example

~~*H* *H* *T* *H* *T* *H*~~

H T H T T T

Use posteriors to update parameters

$$p(\text{heads} \mid \text{obs. H}) = \frac{p(\text{H} \mid \text{heads})p(\text{heads})}{p(\text{H})}$$
$$= \frac{.8 * .6}{.8 * .6 + .6 * .4} \approx 0.667$$

$$p(\text{heads} \mid \text{obs. T}) = \frac{p(\text{T} \mid \text{heads})p(\text{heads})}{p(\text{T})}$$
$$= \frac{.2 * .6}{.2 * .6 + .6 * .4} \approx 0.334$$

Q: Is p(heads | obs. H) + p(heads | obs. T) = 1?

A: No.

Three Coins Example

$$\cancel{H \quad H \quad T \quad H \quad T \quad H}$$

H T H T T T

Use posteriors to update parameters

$$p(\text{heads} \mid \text{obs. H}) = \frac{p(\text{H} \mid \text{heads})\, p(\text{heads})}{p(\text{H})} = \frac{.8 * .6}{.8 * .6 + .6 * .4} \approx 0.667$$

$$p(\text{heads} \mid \text{obs. T}) = \frac{p(\text{T} \mid \text{heads})\, p(\text{heads})}{p(\text{T})} = \frac{.2 * .6}{.2 * .6 + .6 * .4} \approx 0.334$$

(in general, p(heads | obs. H) and p(heads | obs. T) do NOT sum to 1)

fully observed setting

$$p(\text{heads}) = \frac{\#\ \text{heads from penny}}{\#\ \text{total flips of penny}}$$

our setting: partially-observed

$$p(\text{heads}) = \frac{\#\ \textit{expected}\ \text{heads from penny}}{\#\ \text{total flips of penny}}$$

Three Coins Example

~~H H T H T H~~

H T H T T T

Use posteriors to update parameters

$$p(\text{heads} \mid \text{obs. H}) = \frac{p(\text{H} \mid \text{heads})\, p(\text{heads})}{p(\text{H})} = \frac{.8 * .6}{.8 * .6 + .6 * .4} \approx 0.667$$

$$p(\text{heads} \mid \text{obs. T}) = \frac{p(\text{T} \mid \text{heads})\, p(\text{heads})}{p(\text{T})} = \frac{.2 * .6}{.2 * .6 + .6 * .4} \approx 0.334$$

$$p^{(t+1)}(\text{heads}) = \frac{\#\ \textit{expected}\ \text{heads from penny}}{\#\ \text{total flips of penny}} = \frac{\mathbb{E}_{p^{(t)}}[\#\ \textit{expected}\ \text{heads from penny}]}{\#\ \text{total flips of penny}}$$

our setting: partially-observed

Three Coins Example

~~*H H T H T H*~~

H T H T T T

Use posteriors to update parameters

$$p(\text{heads} \mid \text{obs. H}) = \frac{p(\text{H} \mid \text{heads})\,p(\text{heads})}{p(\text{H})} = \frac{.8 * .6}{.8 * .6 + .6 * .4} \approx 0.667$$

$$p(\text{heads} \mid \text{obs. T}) = \frac{p(\text{T} \mid \text{heads})\,p(\text{heads})}{p(\text{T})} = \frac{.2 * .6}{.2 * .6 + .6 * .4} \approx 0.334$$

our setting: partially-observed

$$p^{(t+1)}(\text{heads}) = \frac{\#\ \textit{expected}\ \text{heads from penny}}{\#\ \text{total flips of penny}}$$
$$= \frac{\mathbb{E}_{p^{(t)}}[\#\ \textit{expected}\ \text{heads from penny}]}{\#\ \text{total flips of penny}}$$
$$= \frac{2 * p(\text{heads} \mid \text{obs. H}) + 4 * p(\text{heads} \mid \text{obs.}\,T)}{6}$$
$$\approx 0.444$$

Expectation Maximization (EM)

0. Assume *some* value for your parameters

Two step, iterative algorithm:

1. E-step: count under uncertainty (compute expectations)

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Why does EM work?

X: observed data

Y: unobserved data

$\mathcal{C}(\theta)$ = log-likelihood of complete data (X,Y)

$\mathcal{M}(\theta)$ = marginal log-likelihood of observed data X

$\mathcal{P}(\theta)$ = posterior log-likelihood of incomplete data Y

what do $\mathcal{C}, \mathcal{M}, \mathcal{P}$ look like?

Why does EM work?

X: observed data

Y: unobserved data

$\mathcal{C}(\theta)$ = log-likelihood of complete data (X,Y)

$\mathcal{M}(\theta)$ = marginal log-likelihood of observed data X

$\mathcal{P}(\theta)$ = posterior log-likelihood of incomplete data Y

$$\mathcal{C}(\theta) = \sum_i \log p(x_i, y_i)$$

Why does EM work?

X: observed data

Y: unobserved data

$\mathcal{C}(\theta)$ = log-likelihood of complete data (X,Y)

$\mathcal{M}(\theta)$ = marginal log-likelihood of observed data X

$\mathcal{P}(\theta)$ = posterior log-likelihood of incomplete data Y

$$\mathcal{C}(\theta) = \sum_i \log p(x_i, y_i)$$

$$\mathcal{M}(\theta) = \sum_i \log p(x_i) = \sum_i \log \sum_k p(x_i, y = k)$$

Why does EM work?

X: observed data

Y: unobserved data

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$$\mathcal{P}\ (\theta) = \sum_i \log p(y_i | x_i)$$

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$$p_\theta(Y \mid X) = \frac{p_\theta(X,Y)}{p_\theta(X)} \xrightarrow[\text{algebra}]{} p_\theta(X) = \frac{p_\theta(X,Y)}{p_\theta(Y \mid X)}$$

definition of conditional probability

Why does EM work?

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$$p_\theta(Y \mid X) = \frac{p_\theta(X, Y)}{p_\theta(X)} \quad \Rightarrow \quad p_\theta(X) = \frac{p_\theta(X, Y)}{p_\theta(Y \mid X)}$$

$$\mathcal{C}(\theta) = \sum_i \log p(x_i, y_i) \qquad \mathcal{M}(\theta) = \sum_i \log p(x_i) = \sum_i \log \sum_k p(x_i, y = k) \qquad \mathcal{P}\ (\theta) = \sum_i \log p(y_i | x_i)$$

$$\mathcal{M}(\theta) = \mathcal{C}(\theta) - \mathcal{P}(\theta)$$

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$$p_\theta(Y \mid X) = \frac{p_\theta(X,Y)}{p_\theta(X)} \Longrightarrow p_\theta(X) = \frac{p_\theta(X,Y)}{p_\theta(Y \mid X)}$$

$$\mathcal{M}(\theta) = \mathcal{C}(\theta) - \mathcal{P}(\theta)$$

$$\mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{M}(\theta)|X] = \mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{C}(\theta)|X] - \mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{P}(\theta)|X]$$

take a conditional expectation (why? we'll cover this more in variational inference)

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$$\mathcal{M}(\theta) = \mathcal{C}(\theta) - \mathcal{P}(\theta)$$

$$\mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{M}(\theta)|X] = \mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{C}(\theta)|X] - \mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{P}(\theta)|X]$$

$$\mathcal{M}(\theta) = \mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{C}(\theta)|X] - \mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{P}(\theta)|X]$$

$\mathcal{M}$ already sums over Y

$$\mathcal{M}(\theta) = \sum_i \log p(x_i) = \sum_i \log \sum_k p(x_i, y = k)$$

Why does EM work?

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$\mathcal{C}(\theta)$ = log-likelihood of complete data (X,Y)

$\mathcal{M}(\theta)$ = marginal log-likelihood of observed data X

$\mathcal{P}(\theta)$ = posterior log-likelihood of incomplete data Y

$$\mathcal{M}(\theta) = \mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{C}(\theta)|X] - \mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{P}(\theta)|X]$$

$$\mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{C}(\theta)|X] = \sum_i \sum_k p_{\theta^{(t)}}(y = k\,|x_i) \log p(x_i, y = k)$$

Why does EM work?

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$\mathcal{C}(\theta)$ = log-likelihood of complete data (X,Y)

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$$\mathcal{M}(\theta) = \underbrace{\mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{C}(\theta)|X]}_{Q(\theta,\theta^{(t)})} - \underbrace{\mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{P}(\theta)|X]}_{R(\theta,\theta^{(t)})}$$

Let θ^* be the value that maximizes $Q(\theta, \theta^{(t)})$

Why does EM work?

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Let θ^* be the value that maximizes $Q(\theta, \theta^{(t)})$

$$\mathcal{M}(\theta^*) - \mathcal{M}(\theta^{(t)}) = \left(Q(\theta^*, \theta^{(t)}) - Q(\theta^{(t)}, \theta^{(t)})\right) - \left(R(\theta^*, \theta^{(t)}) - R(\theta^{(t)}, \theta^{(t)})\right)$$

Why does EM work?

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$$\mathcal{M}(\theta) = \underbrace{\mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{C}(\theta)|X]}_{Q(\theta,\theta^{(t)})} - \underbrace{\mathbb{E}_{Y\sim\theta^{(t)}}[\mathcal{P}(\theta)|X]}_{R(\theta,\theta^{(t)})}$$

Let θ^* be the value that maximizes $Q(\theta, \theta^{(t)})$

$$\mathcal{M}(\theta^*) - \mathcal{M}\left(\theta^{(t)}\right) = \underbrace{\left(Q\left(\theta^*, \theta^{(t)}\right) - Q(\theta^{(t)}, \theta^{(t)})\right)}_{\geq 0} - \underbrace{\left(R\left(\theta^*, \theta^{(t)}\right) - R(\theta^{(t)}, \theta^{(t)})\right)}_{\leq 0}$$

≤ 0 *(we'll see why with Jensen's inequality, in variational inference)*

Why does EM work?

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Let θ^* be the value that maximizes $Q(\theta,\theta^{(t)})$

$$\mathcal{M}(\theta^*) - \mathcal{M}\left(\theta^{(t)}\right) = \left(Q\left(\theta^*,\theta^{(t)}\right) - Q(\theta^{(t)},\theta^{(t)})\right) - \left(R\left(\theta^*,\theta^{(t)}\right) - R(\theta^{(t)},\theta^{(t)})\right)$$

$$\mathcal{M}(\theta^*) - \mathcal{M}\left(\theta^{(t)}\right) \geq 0$$

EM does not decrease the marginal log-likelihood

Generalized EM

Partial M step: find a θ that simply increases, rather than *maximizes*, Q

Partial E step: only consider *some* of the variables (an online learning algorithm)

EM has its pitfalls

Objective is not convex → converge to a bad local optimum

Computing expectations can be hard: the E-step could require clever algorithms

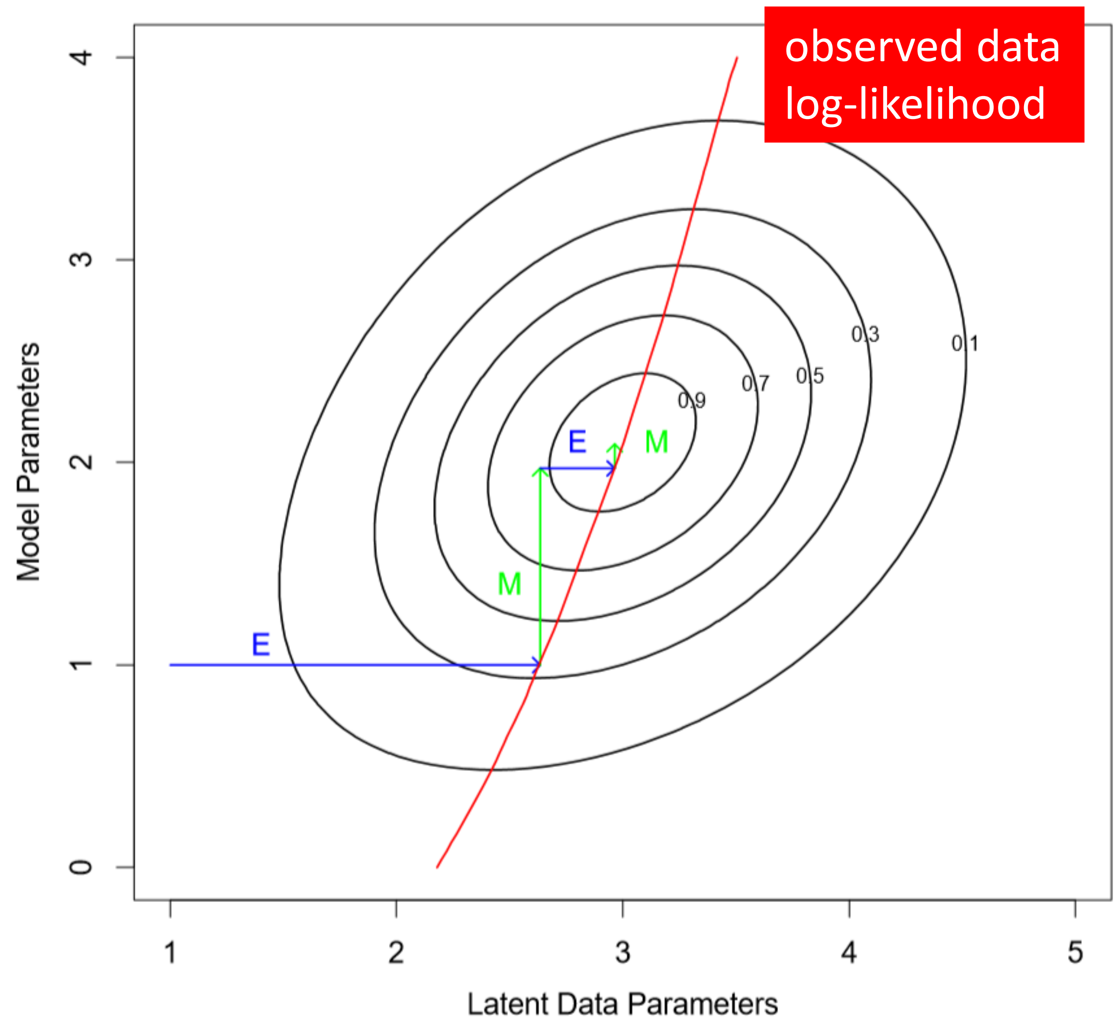
How well does log-likelihood correlate with an end task?

A Maximization-Maximization Procedure

any
distribution
over Z

$$F(\theta, q) = \mathbb{E}[\mathcal{C}(\theta)] - \mathbb{E}[\log q(Z)]$$

we'll see this again with variational inference



Outline

Latent and probabilistic modeling
- Generative Modeling
- Example 1: A Model of Rolling a Die
- Example 2: A Model of Conditional Die Rolls

EM (Expectation Maximization)
- Basic idea
- Three coins example
- Why EM works